LONG-TIME ASYMPTOTICS FOR THE DEGASPERIS-PROCESI EQUATION ON THE HALF-LINE

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Dedicated to the memory of Louis Boutet de Monvel

Abstract. We analyze the long-time asymptotics for the Degasperis-Procesi equation on the half-line. By applying nonlinear steepest descent techniques to an associated 3×3-matrix valued RH problem, we find an explicit formula for the leading order asymptotics of the solution in the similarity region in terms of the initial and boundary values.

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1. Introduction

The nonlinear steepest descent method introduced in [12] provides a powerful technique for determining asymptotics of solutions of nonlinear integrable PDEs. By appropriately deforming the contour of the associated Riemann-Hilbert (RH) problem, the long-time behavior of the solution can be determined by adding up the contributions from the individual critical points. In this way the asymptotics associated with the modified KdV [12], the nonlinear Schrödinger [10], and several other integrable equations posed on the real line have been rigorously established, see [2, 11, 13]. More recently, a number of works treating periodic problems [14] as well as initial-boundary value problems [1, 3] have also appeared.

In this paper we use the method of nonlinear steepest descent to analyze long-time asymptotics for the Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 3\kappa u_x + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \qquad \kappa > 0, \tag{1.1}$$

posed in the domain

$$\Omega = \left\{(x,t) \in \mathbb{R}^2 \,|\, 0 \le x < \infty,\ 0 \le t < \infty\right\}. \tag{1.2}$$

Our main result (see Theorem 5.1 below) gives an exact formula for the leading order asymptotics of $u(x,t)$ in the similarity region $0 < \frac{x}{t} < 3$ in terms of the initial and boundary values. In this region it has the form of slowly decaying oscillations, whereas in the complementary region $\frac{x}{t} > 3$ it is dominated by solitons, if any, see [2, 3, 4].

Equation (1.1) was discovered in [7] using methods of asymptotic integrability. A Lax pair and a bi-Hamiltonian structure were derived in [8]. An interesting aspect of (1.1) is the existence of peaked solutions [8] as well as weak solutions with a very low degree of regularity [5]. The latter class includes a class of discontinuous generalizations of the peakons called shock-peakons [17]. The asymptotic behavior of the solution of (1.1) on the line was determined in [4]. In [15] the solution of the initial-boundary value problem of (1.1) on the half-line was expressed in terms of the solution of a 3×3-matrix RH problem.

Compared with most other applications of the nonlinear steepest descent approach, the asymptotic analysis of (1.1) presents a number of additional difficulties:

(a) The RH problem associated with (1.1) involves 3×3 matrices instead of 2×2 matrices. This implies that the standard uniqueness results for L^2-RH problems (such as Theorem 7.18 of [9]) do not apply. However, it turns out that in an appropriate function space, which we denote by $\dot{L}^3$, uniqueness holds also for 3×3-matrix valued RH problems, see [16]. Thus, by developing the nonlinear steepest descent approach in the $\dot{L}^3$-setting, rigorous asymptotic formulas can still be obtained.

(b) The t-part of the Lax pair associated with (1.1) has singularities at the points $K_j = e^{\frac{\pi i j}{3} - \frac{\pi i}{6}}$, $j = 1, \dots, 6$. In [15] this difficulty was overcome by utilizing two different sets of eigenfunctions which were solutions of two different Lax pairs (a similar idea was used already in [4] to recover $u(x,t)$ for the problem on the line). Here we adopt a similar approach; however, in order to obtain a RH problem suitable for the asymptotic analysis of (1.1), we use a modification of the RH problem in [15]. The modified problem has the advantage that, after the appropriate contour deformations prompted by the nonlinear steepest descent method have been performed, the RH problem involves only one set of eigenfunctions near each of the twelve critical points. This leads to a jump matrix near each critical point of an appropriate form.

(c) The Lax pair associated with (1.1) has singularities at the sixth roots of unity $\varkappa_j = e^{\frac{\pi i (j-1)}{3}}$, $j = 1, \dots, 6$. In [4, 15] this difficulty was overcome by considering a regular RH problem for an associated row vector. Here, rather than trying to develop a nonlinear steepest descent approach for row vector RH problems, we carry out the steepest descent analysis using a regular 3×3-matrix valued solution which, in general, is different from the original solution. However, by uniqueness for the row vector RH problem, the row vectors associated with these two solutions coincide.

(d) On the half-line, the jump contour for the RH problem associated with (1.1) involves nontransversal intersection points, see Figure 1. This implies that the standard theory of L^p-RH problems does not apply. We circumvent this difficulty by employing the theory of L^p-RH problems developed in [16] for general Carleson jump contours.

In Section 2, we give a short review of the RH approach for (1.1) on the half-line. In Section 3, we formulate a RH problem suitable for determining the long-time asymptotics. In Section 4, we prove a nonlinear steepest descent theorem appropriate for analyzing the asymptotics in the similarity region. In Section 5, we prove our main theorem.

2. Preliminaries

We consider initial-boundary value problems for (1.1) for which the initial and boundary values

$$u_0(x) = u(x,0), \qquad x \geq 0, \tag{2.1a}$$

$$g_0(t) = u(0,t), \quad g_1(t) = u_x(0,t), \quad g_2(t) = u_{xx}(0,t), \qquad t \geq 0, \tag{2.1b}$$

satisfy the three conditions

$$u_0(x) - u_{0xx}(x) + \kappa > 0, \qquad x \geq 0, \tag{2.2a}$$

$$g_0(t) - g_2(t) + \kappa > 0, \qquad t \geq 0, \tag{2.2b}$$

$$g_0(t) \leq 0, \qquad t \geq 0. \tag{2.2c}$$

The assumptions in (2.2) imply the following positivity condition which is needed for the spectral analysis:

$$u(x,t) - u_{xx}(x,t) + \kappa > 0, \qquad (x,t) \in \Omega. \tag{2.3}$$

In view of (2.3), we may define $q(x,t)$ by

$$q(x,t) = \big(u(x,t) - u_{xx}(x,t) + \kappa\big)^{\frac{1}{3}}, \qquad (x,t) \in \Omega. \tag{2.4}$$

We next give a short review of the RH approach for (1.1) on the half-line; see [15] for further details. We suppose that $\{g_j\}_0^2$ belong to the Schwartz class $\mathcal{S}(\mathbb{R}_+)$ and that there exists a unique smooth solution $u(x,t)$ of (1.1) in Ω such that (2.1) and (2.2) are satisfied and $u(\cdot,t) \in \mathcal{S}(\mathbb{R}_+)$ for each $t \geq 0$. For simplicity, we henceforth assume that $\kappa = 1$.

2.1. **Lax pairs.** Equation (1.1) admits the Lax pair [4, 6]

$$\begin{cases} \psi_x(x,t,k) = L(x,t,k)\psi(x,t,k), \\ \psi_t(x,t,k) = Z(x,t,k)\psi(x,t,k), \end{cases} \tag{2.5}$$

where $k \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the spectral parameter, $\psi(x,t,k)$ is a 3×3-matrix valued eigenfunction, the 3×3-matrix valued functions L and Z are defined by

$$L(x,t,k) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda q^3 & 1 & 0 \end{pmatrix}, \qquad Z(x,t,k) = \begin{pmatrix} u_x - \frac{2}{3\lambda} & -u & \frac{1}{\lambda} \\ u+1 & \frac{1}{3\lambda} & -u \\ u_x - \lambda u q^3 & 1 & -u_x + \frac{1}{3\lambda} \end{pmatrix},$$

and $\lambda = \lambda(k)$ is defined by

$$\lambda = \frac{1}{3\sqrt{3}}\left(k^3 + \frac{1}{k^3}\right).$$

Let $\omega = e^{\frac{2\pi i}{3}}$. Define $\{l_j\}_1^3$ and $\{z_j\}_1^3$ by

$$l_j(k) = \frac{1}{\sqrt{3}}\left(\omega^j k + \frac{1}{\omega^j k}\right), \quad z_j(k) = \sqrt{3}\left(\frac{(\omega^j k)^2 + (\omega^j k)^{-2}}{k^3 + k^{-3}}\right), \qquad k \in \mathbb{C}. \tag{2.6}$$

Let

$$P(k) = \begin{pmatrix} 1 & 1 & 1 \\ l_1(k) & l_2(k) & l_3(k) \\ l_1^2(k) & l_2^2(k) & l_3^2(k) \end{pmatrix}, \qquad k \in \mathbb{C}, \tag{2.7}$$

and define $\{V_j(x,t,k), \tilde{V}_j(x,t,k)\}_1^2$ by

$$V_1 = P^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda(q^3-1) & 0 & 0 \end{pmatrix} P, \qquad V_2 = P^{-1}\begin{pmatrix} u_x & -u & 0 \\ u & 0 & -u \\ u_x - \lambda u q^3 & 0 & -u_x \end{pmatrix} P,$$

$$\tilde{V}_1 = P^{-1}\begin{pmatrix} \frac{q_x}{q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{q} - q & -\frac{q_x}{q} \end{pmatrix} P,$$

$$\tilde{V}_2 = P^{-1}\left[\begin{pmatrix} -\frac{uq_x}{q} & 0 & 0 \\ \frac{u+1}{q} - 1 & 0 & 0 \\ \frac{u_x}{q^2} & \frac{1}{q} - 1 + uq & \frac{uq_x}{q} \end{pmatrix} + \frac{q^2-1}{\lambda}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] P.$$

Let $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$ and $\mathcal{Z} = \operatorname{diag}(z_1, z_2, z_3)$. The eigenfunctions Φ and $\tilde{\Phi}$ introduced by

$$\psi(x,t,k) = P(k)\Phi(x,t,k)e^{\mathcal{L}(k)x+\mathcal{Z}(k)t}, \tag{2.8a}$$

$$\psi(x,t,k) = D(x,t)P(k)\tilde{\Phi}(x,t,k)e^{\mathcal{L}(k)y(x,t)+\mathcal{Z}(k)t}, \tag{2.8b}$$

where

$$y(x,t) = \int_{(0,0)}^{(x,t)} q(x',t')\left(dx' - u(x',t')dt'\right), \qquad D(x,t) = \begin{pmatrix} \frac{1}{q(x,t)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q(x,t) \end{pmatrix}, \tag{2.9}$$

satisfy the Lax pair equations

$$\begin{cases} \Phi_x - [\mathcal{L}, \Phi] = V_1\Phi, \\ \Phi_t - [\mathcal{Z}, \Phi] = V_2\Phi, \end{cases} \tag{2.10a}$$

and

$$\begin{cases} \tilde{\Phi}_x - [q\mathcal{L}, \tilde{\Phi}] = \tilde{V}_1\tilde{\Phi}, \\ \tilde{\Phi}_t - [\mathcal{Z} - uq\mathcal{L}, \tilde{\Phi}] = \tilde{V}_2\tilde{\Phi}, \end{cases} \tag{2.10b}$$

respectively.

2.2. **Analytic eigenfunctions.** Let γ_j, $j = 1,2,3$, denote contours in the (x,t)-plane connecting (x_j, t_j) with (x,t), where $(x_1,t_1) = (0,\infty)$, $(x_2,t_2) = (0,0)$, and $(x_3,t_3) = (\infty, t)$. The contours can be chosen to consist of straight line segments parallel to the x- or t-axis. For a diagonal matrix D, let $\hat{D}$ denote the operator which acts on a matrix A by $\hat{D}A = [D,A]$, i.e. $e^{\hat{D}}A = e^{D}Ae^{-D}$. We define solutions $\{\Phi_n(x,t,k)\}_1^{18}$ and $\{\tilde{\Phi}_n(x,t,k)\}_1^{18}$ of the Lax pairs (2.10a) and (2.10b) respectively, by the solutions of the integral equations

$$(\Phi_n)_{ij}(x,t,k) = \delta_{ij} + \int_{\gamma_{ij}^n} \left(e^{\hat{\mathcal{L}}(k)x+\hat{\mathcal{Z}}(k)t}W_n(x',t',k)\right)_{ij}, \qquad k \in D_n, \quad i,j = 1,2,3, \tag{2.11a}$$

$$(\tilde{\Phi}_n)_{ij}(x,t,k) = \delta_{ij} + \int_{\gamma_{ij}^n} \left(e^{\hat{\mathcal{L}}(k)y(x,t)+\hat{\mathcal{Z}}(k)t}\tilde{W}_n(x',t',k)\right)_{ij}, \qquad k \in D_n, \quad i,j = 1,2,3, \tag{2.11b}$$

where the contours γ_{ij}^n, $n = 1,\dots,18$, $i,j = 1,2,3$, are given by

$$\gamma_{ij}^n = \begin{cases} \gamma_1, & \operatorname{Re} l_i(k) < \operatorname{Re} l_j(k), \quad \operatorname{Re} z_i(k) \geq \operatorname{Re} z_j(k), \\ \gamma_2, & \operatorname{Re} l_i(k) < \operatorname{Re} l_j(k), \quad \operatorname{Re} z_i(k) < \operatorname{Re} z_j(k), \\ \gamma_3, & \operatorname{Re} l_i(k) \geq \operatorname{Re} l_j(k), \end{cases} \qquad \text{for} \quad k \in D_n, \tag{2.12}$$

the closed one-forms $W_n(x,t,k)$, $\tilde{W}_n(x,t,k)$ are defined by

$$W_n = e^{-\hat{\mathcal{L}}x-\hat{\mathcal{Z}}t}(V_1dx + V_2dt)\Phi_n, \qquad \tilde{W}_n = e^{-\hat{\mathcal{L}}y-\hat{\mathcal{Z}}t}(\tilde{V}_1dx + \tilde{V}_2dt)\tilde{\Phi}_n,$$

and the open sets $\{D_n\}_1^{18}$ are displayed in Figure 1 (precise definitions of the D_n's are given in [15]).

Let $K_j = e^{\frac{\pi ij}{3} - \frac{\pi i}{6}}$, $j = 1,\dots,6$, denote the points where $\lambda = 0$ and let $\varkappa_j = e^{\frac{\pi i(j-1)}{3}}$, $j = 1,\dots,6$, denote the sixth roots of unity, see Figure 2. Away from the sets $\{\infty, 0\} \cup \{\varkappa_j\}_1^6 \cup \{k_j\}$ and $\{\varkappa_j, K_j\}_1^6 \cup \{k_j\}$, respectively, Φ_n and $\tilde{\Phi}_n$ are bounded and analytic functions of $k \in D_n$ with continuous extensions to $\bar{D}_n$. Here $\{k_j\}$ denotes a possibly empty set of singularities at which the Fredholm determinant of integral

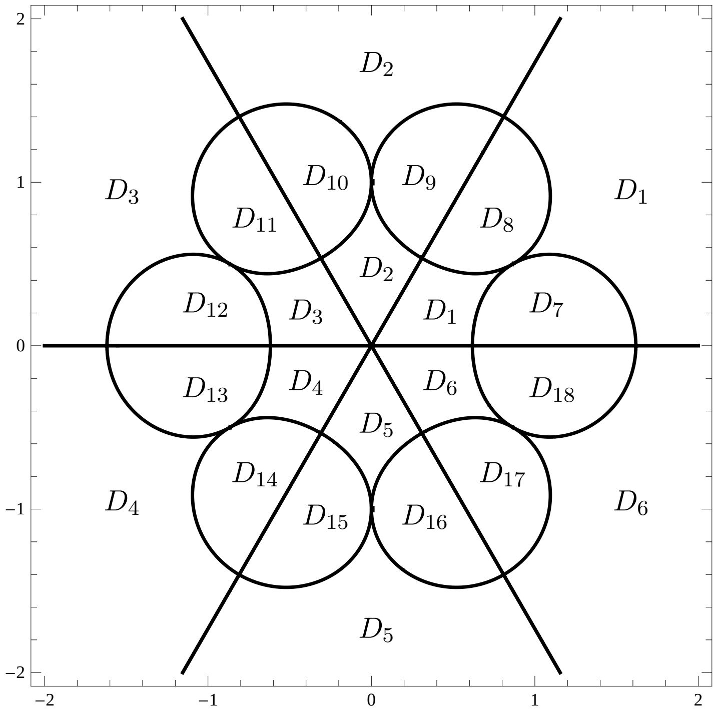

Figure 1. *The sets D_n, $n = 1, \dots, 18$, which decompose the complex k-plane.*

equations (2.11) vanishes; for simplicity, we henceforth assume that the set $\{k_j\}$ is empty (solitonless case). For those n for which the indicated limiting points lie on the boundary of the corresponding D_n,

$$\begin{aligned}
&\Phi_n(x,t,k) = I + O(k - K_j) \quad \text{as} \quad k \to K_j, \quad k \in D_n, \qquad j = 1, \dots, 6,\\
&\tilde{\Phi}_n(x,t,k) = I + O(1/k) \quad \text{as} \quad k \to \infty, \quad k \in D_n,\\
&\tilde{\Phi}_n(x,t,k) = I + O(k) \quad \text{as} \quad k \to 0, \quad k \in D_n,
\end{aligned}$$

where I denotes the identity matrix.

We define spectral functions $\{S_n(k)\}_1^{18}$ and $\{\tilde{S}_n(k)\}_1^{18}$ by

$$S_n(k) = \Phi_n(0,0,k), \qquad \tilde{S}_n(k) = \tilde{\Phi}_n(0,0,k), \qquad k \in D_n. \tag{2.13}$$

2.3. **Symmetries.** Define sectionally analytic functions $S_*(k)$ and $\tilde{S}_*(k)$ for $k \in \mathbb{C}$ by setting $S_*(k) = S_n(k)$ and $\tilde{S}_*(k) = \tilde{S}_n(k)$ for $k \in D_n$. If F denotes one of the 3×3-matrix valued functions $\mathcal{L}$, $\mathcal{Z}$, M, S_*, or $\tilde{S}_*$, then F obeys the symmetries

$$F(k) = \mathcal{A} F(\omega k) \mathcal{A}^{-1}, \qquad k \in \mathbb{C}, \tag{2.14a}$$

$$F(k) = \mathcal{B} F(1/k) \mathcal{B}, \qquad k \in \mathbb{C}, \tag{2.14b}$$

$$F(k) = \mathcal{B} \overline{F(\overline{k})} \mathcal{B}, \qquad k \in \mathbb{C}, \tag{2.14c}$$

where $\mathcal{A}$, $\mathcal{B}$ are defined by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.15}$$

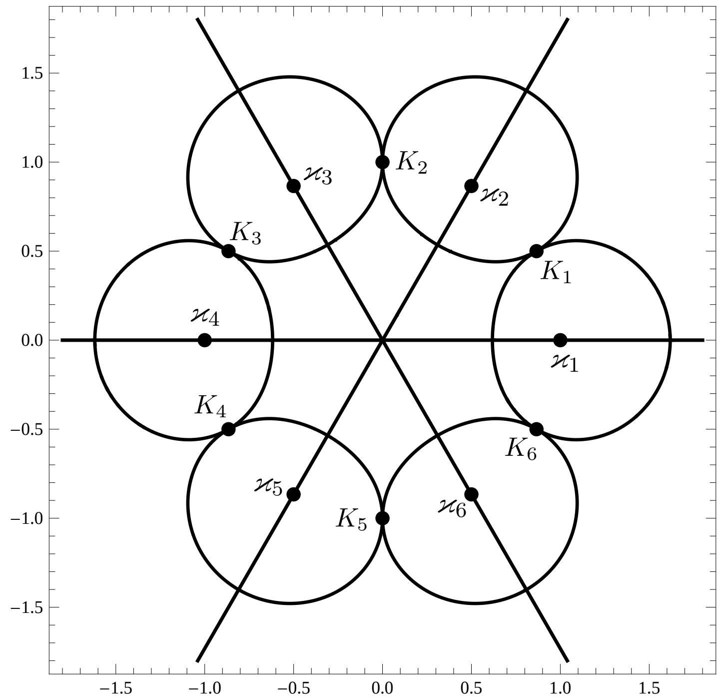

Figure 2. *The points* $K_j = e^{\frac{\pi i j}{3} - \frac{\pi i}{6}}$, $j = 1, \dots, 6$, *where* $\lambda = 0$, *and the points* $\varkappa_j = e^{\frac{\pi i (j-1)}{3}}$, $j = 1, \dots, 6$, *where* $P^{-1}(k)$ *has poles.*

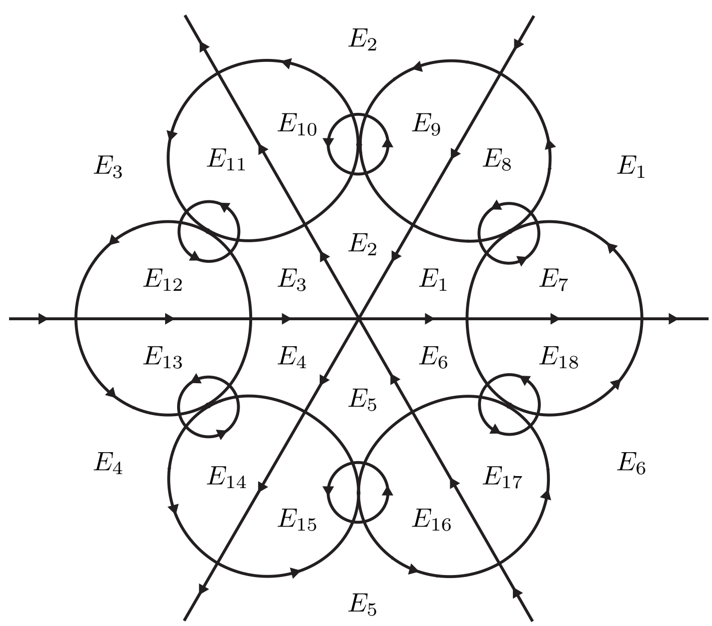

Figure 3. *The sets* E_n *which decompose the complex* k*-plane.*

3. A Riemann-Hilbert problem

We use the eigenfunctions Φ_n and $\tilde{\Phi}_n$ to define a RH problem suitable for analyzing the long-time asymptotics.

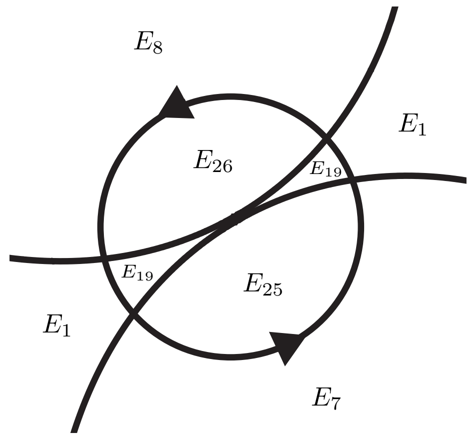

Figure 4. *The sets E_n for k near K_1.*

Choose a small radius $r > 0$ and let U_j denote the open disk of radius r centered at K_j, $j = 1, \dots, 6$. Let $U = \cup_{j=1}^6 U_j$ and define open sets $\{E_n\}_1^{36}$ by, see Figures 3 and 4:

$$E_n = D_n \setminus \bar{U}, \qquad E_{n+18} = D_n \cap U, \qquad n = 1, \dots, 18.$$

The eigenfunctions $\{\tilde{\Phi}_n\}_1^{18}$ are well-behaved near $k = \infty$ and $k = 0$ while the eigenfunctions $\{\Phi_n\}_1^{18}$ are well-behaved near the K_j's. We formulate a RH problem relative to the contour shown in Figure 3 (see also Figure 4) by using $\tilde{\Phi}_n$ and Φ_n for k in E_n and E_{n+18}, respectively.

Let $y = y(x,t)$ be the function defined in (2.9). The map $F : (x,t) \mapsto (y,t)$ is a bijection from $\Omega = \{x \geq 0, t \geq 0\}$ onto $F(\Omega) \subset \mathbb{R}^2$. Thus, for each $(y,t) \in F(\Omega)$, we may define a sectionally meromorphic function $M(y,t,k)$ by

$$M(y,t,k) = \begin{cases} \tilde{\Phi}_n(x,t,k), & k \in E_n, \\ P(k)^{-1}D(x,t)^{-1}P(k)\Phi_n(x,t,k)e^{(x-y+\nu_0)\mathcal{L}(k)}, & k \in E_{n+18}, \end{cases} \tag{3.1}$$

where $n = 1, \dots, 18$ and the constant $\nu_0 \in \mathbb{R}$ is defined by

$$\nu_0 = \lim_{x\to\infty}(y - x) = \int_0^\infty (q(x,0) - 1)dx.$$

Let M_n denote the restriction of M to E_n. The definition (3.1) and the relations (2.8) imply that M satisfies the jump condition

$$M_n = M_m J_{m,n}, \qquad k \in \bar{E}_n \cap \bar{E}_m, \tag{3.2}$$

where

$$\begin{cases} J_{m,n}(y,t,k) = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}}\big(\tilde{S}_m^{-1}(k)\tilde{S}_n(k)\big), \\ J_{n,n+18}(y,t,k) = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}}C_n(k), \\ J_{m+18,n+18}(y,t,k) = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}}e^{-\nu_0\hat{\mathcal{L}}}\big(S_m^{-1}(k)S_n(k)\big), \end{cases} \qquad n,m = 1,\dots,18. \tag{3.3}$$

The functions $\{C_n(k)\}_1^{18}$ are defined as follows. By (2.8) the functions

$$\psi_n = DP\tilde{\Phi}_n e^{y\mathcal{L}+t\mathcal{Z}} = DPM_n e^{y\mathcal{L}+t\mathcal{Z}},$$
$$\psi_{n+18} = P\Phi_n e^{x\mathcal{L}+t\mathcal{Z}} = DPM_{n+18}e^{y\mathcal{L}+t\mathcal{Z}-\nu_0\mathcal{L}}$$

solve the same differential equations (2.5), hence $\psi_{n+18} = \psi_n\tilde{C}_n(k)$ with $\tilde{C}_n(k)$ independent of (x,t). Thus, $M_{n+18} = M_n e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}}C_n(k)$ with $C_n(k) = \tilde{C}_n(k)e^{\nu_0\mathcal{L}}$, and

$C_n(k) = \psi_n^{-1}\psi_{n+18}e^{\nu_0\mathcal{L}}$ satisfies

$$C_n(k) = e^{-y\mathcal{L}-t\mathcal{Z}}\tilde{\Phi}_n(x,t,k)^{-1}P(k)^{-1}D(x,t)^{-1}P(k)\Phi_n(x,t,k)e^{x\mathcal{L}+t\mathcal{Z}}e^{\nu_0\mathcal{L}}. \tag{3.4}$$

Proposition 3.1. *Let $E = \cup_{n=1}^{36} E_n$ and let $(y,t) \in F(\Omega)$. Except for possible singularities at the points $\{\varkappa_j\}_1^6$, $M(y,t,k)$ is a bounded and analytic function of $k \in E$. Moreover,*

$$M(x,t,k) = I + O(1/k) \quad \text{uniformly as} \quad k \to \infty, \quad k \in \mathbb{C}.$$

Proof. Since $P(k)^{-1}D(x,t)^{-1}P(k)$ is an analytic function of $k \in \hat{\mathbb{C}}$ except for poles at the points $\varkappa_j$, the result follows immediately from the properties of the functions Φ_n and $\tilde{\Phi}_n$. □

The singularity structure of M_n at the $\varkappa_j$'s implies that the function N defined by

$$N(y,t,k) = (1,1,1)M(y,t,k), \qquad k \in E, \tag{3.5}$$

is nonsingular at the $\varkappa_j$'s, see [4, 15]. Together with Proposition 3.1 and Lemma A.3, this implies that, for each $(y,t) \in F(\Omega)$,

$$N(y,t,\cdot) \in (1,1,1) + \dot{E}^3(E) \cap E^\infty(E),$$

where the function spaces $\dot{E}^3(E)$ and $E^\infty(E)$ are defined in Appendix A. The following result follows.

Proposition 3.2. *For each $(y,t) \in F(\Omega)$, the function $N(y,t,\cdot)$ is a row vector solution of the L^3-RH problem*

$$\begin{cases} N(y,t,\cdot) \in (1,1,1) + \dot{E}^3(E), \\ N_n(y,t,k) = N_m(y,t,k)J_{m,n}(y,t,k) \quad \text{for a.e. } k \in \bar{E}_n \cap \bar{E}_m, \quad n,m = 1,\dots,36. \end{cases} \tag{3.6}$$

Remark 3.3. Although the RH problem formulated in (3.6) differs from the one used in [15], both problems rely on the same idea of using the Φ_n's near the K_j's and the $\tilde{\Phi}_n$'s near $\{0,\infty\}$. The RH problem formulated in (3.6) is better adapted for our present purposes because it uses only one set of eigenfunctions, namely the $\tilde{\Phi}_n$'s, near the three lines $\mathbb{R}$, $\omega\mathbb{R}$, and $\omega^2\mathbb{R}$.

3.1. **Jump matrix.** Define functions $\{r(k), h(k), \check{r}(k), \check{h}(k)\}$ by

$$\begin{aligned} r(k) &= (\tilde{S}_{18}(k)^{-1}\tilde{S}_7(k))_{21}, \qquad k \in \bar{E}_7 \cap \bar{E}_{18}, \\ h(k) &= (\tilde{S}_6(k)^{-1}\tilde{S}_{18}(k))_{21}, \qquad k \in \bar{E}_6 \cap \bar{E}_{18}, \\ \check{r}(k) &= (\tilde{S}_{13}(k)^{-1}\tilde{S}_{12}(k))_{21}, \qquad k \in \bar{E}_{12} \cap \bar{E}_{13}, \\ \check{h}(k) &= (\tilde{S}_4(k)^{-1}\tilde{S}_{13}(k))_{21}, \qquad k \in \bar{E}_4 \cap \bar{E}_{13}. \end{aligned}$$

Proceeding as in Section 5 of [15] we can express the matrices $\{\tilde{S}_n\}_1^{18}$ in terms of the entries of only two 3×3-matrix valued functions $\tilde{S}(k)$ and $\tilde{s}(k)$ (see Proposition 5.1 in [15]). In terms of these entries we have

$$\begin{aligned} r(k) &= \frac{\tilde{s}_{11}m_{12}(\tilde{S}) - \tilde{s}_{21}m_{22}(\tilde{S}) + \tilde{s}_{31}m_{32}(\tilde{S})}{\tilde{s}_{12}m_{12}(\tilde{S}) - \tilde{s}_{22}m_{22}(\tilde{S}) + \tilde{s}_{32}m_{32}(\tilde{S})}, \\ h(k) &= \frac{-m_{33}(\tilde{s})m_{12}(\tilde{S}) + m_{13}(\tilde{s})m_{32}(\tilde{S})}{\tilde{s}_{22}(\tilde{s}_{12}m_{12}(\tilde{S}) - \tilde{s}_{22}m_{22}(\tilde{S}) + \tilde{s}_{32}m_{32}(\tilde{S}))}, \end{aligned}$$

where $m_{ij}(\tilde{s})$ and $m_{ij}(\tilde{S})$ denote the (ij)th minors of the matrices $\tilde{s}(k)$ and $\tilde{S}(k)$ respectively. Simplification shows that the sum $r+h$ has an analytic continuation to $\bar{E}_1 \cap \bar{E}_6$ which satisfies

$$r(k) + h(k) = \frac{\tilde{s}_{21}(k)}{\tilde{s}_{22}(k)} = (\tilde{S}_6(k)^{-1}\tilde{S}_1(k))_{21}, \qquad k \in \bar{E}_1 \cap \bar{E}_6.$$

Similarly, the sum $\check{r} + \check{h}$ has an analytic continuation to $\bar{E}_3 \cap \bar{E}_4$ which satisfies

$$\check{r}(k) + \check{h}(k) = \frac{m_{12}(\tilde{s}(k))}{m_{11}(\tilde{s}(k))} = (\tilde{S}_4(k)^{-1}\tilde{S}_3(k))_{21}, \qquad k \in \bar{E}_3 \cap \bar{E}_4.$$

In summary, the approach of Section 5 of [15] shows that with the contour oriented as in Figure 3 the jump matrix for the RH problem (3.6) is given for k near $\mathbb{R}$ by

$$J = \begin{cases} J_{6,1} = \begin{pmatrix} 1 & -(\overline{r(\bar{k})} + \overline{h(\bar{k})})e^{-t\Phi} & 0 \\ (r(k)+h(k))e^{t\Phi} & 1 - |r(k)+h(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_1 \cap \bar{E}_6, \\ J_{18,7} = \begin{pmatrix} 1 & -\overline{r(\bar{k})}e^{-t\Phi} & 0 \\ r(k)e^{t\Phi} & 1-|r(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_7 \cap \bar{E}_{18}, \\ J_{1,7} = \begin{pmatrix} 1 & \overline{h(\bar{k})}e^{-t\Phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_1 \cap \bar{E}_7, \\ J_{6,18} = \begin{pmatrix} 1 & 0 & 0 \\ h(k)e^{t\Phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_6 \cap \bar{E}_{18}, \\ J_{4,3} = \begin{pmatrix} 1 & -(\overline{\check{r}(\bar{k})} + \overline{\check{h}(\bar{k})})e^{-t\Phi} & 0 \\ (\check{r}(k)+\check{h}(k))e^{t\Phi} & 1 - |\check{r}(k)+\check{h}(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_3 \cap \bar{E}_4, \\ J_{13,12} = \begin{pmatrix} 1 & -\overline{\check{r}(\bar{k})}e^{-t\Phi} & 0 \\ \check{r}(k)e^{t\Phi} & 1-|\check{r}(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_{12} \cap \bar{E}_{13}, \\ J_{3,12} = \begin{pmatrix} 1 & \overline{\check{h}(\bar{k})}e^{-t\Phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_3 \cap \bar{E}_{12}, \\ J_{4,13} = \begin{pmatrix} 1 & 0 & 0 \\ \check{h}(k)e^{t\Phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_4 \cap \bar{E}_{13}. \end{cases} \tag{3.7}$$

where $\Phi = \Phi(\zeta, k)$ with $\zeta = y/t$ and

$$\Phi(\zeta, k) = (l_2(k) - l_1(k))\zeta + (z_2(k) - z_1(k)). \tag{3.8}$$

The matrices $J_{1,2}$, $J_{5,6}$ and $J_{3,2}$, $J_{4,5}$ are missing in (3.7) because they can be recovered by symmetries from $J_{3,4}$ and $J_{1,6}$, respectively. From (2.14b) and (2.14c), we infer that $r(k^{-1}) = \overline{r(\bar{k})}$ and $h(k^{-1}) = \overline{h(\bar{k})}$. In particular, $|r(k^{-1})| = |r(k)|$ for $k \in \bar{E}_7 \cap \bar{E}_{18}$. The functions $\check{r}$ and $\check{h}$ satisfy similar symmetries.

In a similar way we find that the jump matrix J for k near K_1 is given by

$$J = \begin{cases} J_{19,25} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & f_1(k) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_{19} \cap \bar{E}_{25}, \\ J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f_2(k) \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_{19} \cap \bar{E}_{26}, \end{cases} \tag{3.9}$$

where the functions f_1 and f_2 are bounded and continuous on the given subcontours.

We finally need the form of the jump matrix J for k on the circles where the E_n's and E_{n+18}'s meet.

Lemma 3.4. *With the contour oriented as in Figure 3, we have*

$$J = \begin{cases} J_{1,19} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & g_1(k) & g_2(k) \\ 0 & 1 & g_3(k) \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_1 \cap \bar{E}_{19}, \\ J_{7,25} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & 0 & g_4(k) \\ 0 & 1 & g_5(k) \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_7 \cap \bar{E}_{25}, \\ J_{8,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & g_6(k) & g_7(k) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{E}_8 \cap \bar{E}_{26}, \end{cases} \tag{3.10}$$

where the functions $\{g_j(k)\}_1^7$ are bounded and continuous on the given subcontours.

Proof. In view of (3.3), it is enough to show that

$$\begin{aligned} &C_1(k) = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad k \in \bar{D}_1; \qquad C_7(k) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad k \in \bar{D}_7, \\ &C_8(k) = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad k \in \bar{D}_8, \end{aligned} \tag{3.11}$$

where $*$ denotes an entry which is bounded and continuous except for possible singularities at the points $\varkappa_j$, K_j, 0, and ∞. For $n = 1, 7, 8$, the matrices $(\gamma^n)_{ij} = \gamma^n_{ij}$ are given by (see (2.12))

$$\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_2 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}, \qquad \gamma^7 = \begin{pmatrix} \gamma_3 & \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}, \qquad \gamma^8 = \begin{pmatrix} \gamma_3 & \gamma_2 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}.$$

Hence evaluation of (3.4) as $(y,t) \to (\infty, 0)$ yields

$$C_n(k) = \lim_{y\to\infty} e^{-\nu_0\mathcal{L}} e^{-\mathcal{L}x} \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} e^{\mathcal{L}y} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad k \in D_n,$$

for $n = 1, 7, 8$. Moreover, thanks to the assumed decay of the Dirichlet and Neumann values as $t \to \infty$, the functions $\tilde{\Phi}_n$ and Φ_n are bounded as $(y,t) \to (0,\infty)$ for each $k \in D_n$. Consequently, using that $\operatorname{Re} z_2 < \operatorname{Re} z_1 < \operatorname{Re} z_3$ in D_7 and $\operatorname{Re} z_1 < \operatorname{Re} z_3 <$

$\operatorname{Re} z_2$ in D_8, evaluation of (3.4) as $(y,t) \to (0,\infty)$ yields $(C_7(k))_{12} = 0$ for $k \in D_7$ and $(C_8(k))_{23} = 0$ for $k \in D_8$. This proves (3.11). $\square$

4. A nonlinear steepest descent theorem

We prove a nonlinear steepest descent theorem suitable for determining the asymptotics of (1.1) in the similarity region.

For $r > 0$, let $X^r = X_1^r \cup \cdots \cup X_4^r$ denote the cross $X = X_1 \cup \cdots \cup X_4$ defined in (B.1) restricted to the disk of radius r centered at the origin, i.e. $X^r = X \cap \{|z| < r\}$. The spaces $\dot{E}^p$ and $\dot{L}^p$ are defined in Appendix A.

Theorem 4.1. *Let $\mathcal{I} \subset \mathbb{R}$ be a (possibly infinite) interval. Let $\rho, \epsilon : \mathcal{I} \to (0,\infty)$ be bounded strictly positive functions. Let $k_0 : \mathcal{I} \to [1/2, 1)$ be a function such that $k_0(\zeta) + \epsilon(\zeta) < 1$ and $\epsilon(\zeta) < k_0(\zeta)/2$ for each $\zeta \in \mathcal{I}$. We henceforth drop the ζ dependence of these functions and write simply ρ, ϵ, k_0 for $\rho(\zeta)$, $\epsilon(\zeta)$, $k_0(\zeta)$, respectively.*

Let $\Gamma = \Gamma(\zeta)$ be a family of Carleson jump contours parametrized by $\zeta \in \mathcal{I}$ such that:

- (Γ1) *For each $\zeta \in \mathcal{I}$, Γ contains the small crosses $\pm k_0 + X^\epsilon$ as a subset.*
- (Γ2) *For each $\zeta \in \mathcal{I}$, Γ is invariant as a set under the maps*
$$k \mapsto \omega k, \qquad k \mapsto 1/k. \tag{4.1}$$
Moreover, the orientation of Γ is such that if k traverses Γ in the positive direction, then ωk and $1/k$ also traverse Γ in the positive direction.
- (Γ3) *Let $\mathcal{V}$ denote the union of the two disks $\{|k \pm k_0| < \epsilon\}$ and the sets obtained by letting the symmetries in (4.1) act repeatedly on these disks. Let $\hat{\Gamma} = \Gamma \cup \partial\mathcal{V}$ and assume that the boundary of each of the 12 components of $\mathcal{V}$ is oriented counterclockwise. Then, after reversing the orientation on a subcontour if necessary, $\hat{\Gamma}$ is a Carleson jump contour for each $\zeta \in \mathcal{I}$.*
- (Γ4) *The contour remains a bounded distance away from the point $K_1 := e^{\frac{i\pi}{6}}$ for all $\zeta \in \mathcal{I}$:*
$$\inf_{\zeta \in \mathcal{I}} \operatorname{dist}(K_1, \hat{\Gamma}) > 0. \tag{4.2}$$

We also assume that the Cauchy singular operator $\mathcal{S}_{\hat{\Gamma}}$ defined by
$$(\mathcal{S}_{\hat{\Gamma}} h)(z) = \lim_{r \to 0} \frac{1}{\pi i} \int_{\hat{\Gamma} \setminus \{|z'-z| < r\}} \frac{h(z')}{z' - z} dz',$$
is uniformly[1] *bounded on $L^2(\hat{\Gamma})$, i.e.*
$$\sup_{\zeta \in \mathcal{I}} \|\mathcal{S}_{\hat{\Gamma}}\|_{\mathcal{B}(L^2(\hat{\Gamma}))} < \infty. \tag{4.3}$$

Consider the following family of L^3-RH problems parametrized by the two parameters $\zeta \in \mathcal{I}$ and $t > 0$:
$$\begin{cases} m(\zeta, t, \cdot) \in I + \dot{E}^3(\hat{\mathbb{C}} \setminus \Gamma), \\ m_+(\zeta, t, k) = m_-(\zeta, t, k) v(\zeta, t, k) \quad \text{for a.e. } k \in \Gamma, \end{cases} \tag{4.4}$$
where the jump matrix $v(\zeta, t, k)$ obeys the symmetries (2.14a) and (2.14b) and satisfies
$$w(\zeta, t, \cdot) := v(\zeta, t, \cdot) - I \in \dot{L}^1(\Gamma) \cap L^\infty(\Gamma), \qquad \zeta \in \mathcal{I}, \quad t > 0. \tag{4.5}$$

[1] For any fixed $\zeta \in \mathcal{I}$, $\mathcal{S}_{\hat{\Gamma}}$ is bounded on $L^2(\hat{\Gamma})$ as a consequence of (Γ3).

Let $\tau := t\rho^2$. Let Γ_X denote the union of the two small crosses $\pm k_0 + X^\epsilon$ and the sets obtained by letting the symmetries in (4.1) act repeatedly on these crosses. Let $\Gamma' = \Gamma \setminus \Gamma_X$ and suppose

$$\|w(\zeta,t,\cdot)\|_{\dot{L}^p(\Gamma')} = O(\epsilon^{\frac{1}{p}}\tau^{-1}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \quad p\in\{1,\tfrac{3}{2},3\}, \tag{4.6a}$$

$$\|w(\zeta,t,\cdot)\|_{L^\infty(\Gamma')} = O(\tau^{-1}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \tag{4.6b}$$

uniformly with respect to $\zeta\in\mathcal{I}$. Moreover, let $\mathcal{C} = \operatorname{diag}(1,-1,1)$ and suppose that the normalized jump matrices

$$\begin{cases} v_0(\zeta,t,z) = \mathcal{C}v\Big(\zeta,t,k_0 - \frac{\epsilon z}{\rho}\Big)\mathcal{C}, \\ \check{v}_0(\zeta,t,z) = \mathcal{C}\overline{v\Big(\zeta,t,\overline{-k_0+\frac{\epsilon z}{\rho}}\Big)}\mathcal{C}, \end{cases} \qquad z\in X^\rho, \quad \zeta\in\mathcal{I}, \tag{4.7}$$

have the form

$$v_0(\zeta,t,z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ R_1(\zeta,t,z)z^{-2i\nu(\zeta)}e^{t\phi(\zeta,z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_1^\rho, \\ \begin{pmatrix} 1 & -R_2(\zeta,t,z)z^{2i\nu(\zeta)}e^{-t\phi(\zeta,z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_2^\rho, \\ \begin{pmatrix} 1 & 0 & 0 \\ -R_3(\zeta,t,z)z^{-2i\nu(\zeta)}e^{t\phi(\zeta,z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_3^\rho, \\ \begin{pmatrix} 1 & R_4(\zeta,t,z)z^{2i\nu(\zeta)}e^{-t\phi(\zeta,z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_4^\rho, \end{cases} \tag{4.8a}$$

and

$$\check{v}_0(\zeta,t,z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ \check{R}_1(\zeta,t,z)z^{-2i\check{\nu}(\zeta)}e^{t\phi(\zeta,z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_1^\rho, \\ \begin{pmatrix} 1 & -\check{R}_2(\zeta,t,z)z^{2i\check{\nu}(\zeta)}e^{-t\phi(\zeta,z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_2^\rho, \\ \begin{pmatrix} 1 & 0 & 0 \\ -\check{R}_3(\zeta,t,z)z^{-2i\check{\nu}(\zeta)}e^{t\phi(\zeta,z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_3^\rho, \\ \begin{pmatrix} 1 & \check{R}_4(\zeta,t,z)z^{2i\check{\nu}(\zeta)}e^{-t\phi(\zeta,z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z\in X_4^\rho, \end{cases} \tag{4.8b}$$

where:

- *The phase $\phi(\zeta,z)$ is a smooth function of $(\zeta,z)\in\mathcal{I}\times\mathbb{C}$ such that*

$$\phi(\zeta,0)\in i\mathbb{R}, \qquad \frac{\partial\phi}{\partial z}(\zeta,0) = 0, \qquad \frac{\partial^2\phi}{\partial z^2}(\zeta,0) = i, \qquad \zeta\in\mathcal{I}, \tag{4.9}$$

and

$$\operatorname{Re}\phi(\zeta,z) \le -\frac{|z|^2}{4}, \qquad z \in X_1^\rho \cup X_3^\rho, \quad \zeta \in \mathcal{I}, \tag{4.10a}$$

$$\operatorname{Re}\phi(\zeta,z) \ge \frac{|z|^2}{4}, \qquad z \in X_2^\rho \cup X_4^\rho, \quad \zeta \in \mathcal{I}, \tag{4.10b}$$

$$\left|\phi(\zeta,z) - \phi(\zeta,0) - \frac{iz^2}{2}\right| \le C\frac{|z|^3}{\rho}, \qquad z \in X^\rho, \quad \zeta \in \mathcal{I}, \tag{4.10c}$$

where $C > 0$ is a constant.

- *There exist smooth functions $q, \check{q} : \mathcal{I} \to \mathbb{C}$ and constants $(\alpha, L) \in [\frac{1}{3}, 1) \times (0, \infty)$ such that*

$$\sup_{\zeta\in\mathcal{I}} |q(\zeta)| < 1, \qquad \sup_{\zeta\in\mathcal{I}} |\check{q}(\zeta)| < 1,$$

$$\begin{cases} |R_1(\zeta,t,z) - q(\zeta)| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_1^\rho, \\ |R_2(\zeta,t,z) - \frac{\overline{q(\zeta)}}{1-|q(\zeta)|^2}| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_2^\rho, \\ |R_3(\zeta,t,z) - \frac{q(\zeta)}{1-|q(\zeta)|^2}\Big| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_3^\rho, \\ |R_4(\zeta,t,z) - \overline{q(\zeta)}| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_4^\rho, \end{cases} \quad \zeta \in \mathcal{I}, \quad t > 0, \tag{4.11a}$$

and

$$\begin{cases} |\check{R}_1(\zeta,t,z) - \check{q}(\zeta)| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_1^\rho, \\ |\check{R}_2(\zeta,t,z) - \frac{\overline{\check{q}(\zeta)}}{1-|\check{q}(\zeta)|^2}| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_2^\rho, \\ |\check{R}_3(\zeta,t,z) - \frac{\check{q}(\zeta)}{1-|\check{q}(\zeta)|^2}\Big| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_3^\rho, \\ |\check{R}_4(\zeta,t,z) - \overline{\check{q}(\zeta)}| \le L\left|\frac{z}{\rho}\right|^\alpha e^{\frac{t|z|^2}{6}}, & z \in X_4^\rho, \end{cases} \quad \zeta \in \mathcal{I}, \quad t > 0. \tag{4.11b}$$

- *The functions $\nu(\zeta)$ and $\check{\nu}(\zeta)$ are defined by*

$$\nu(\zeta) = -\frac{1}{2\pi}\log(1 - |q(\zeta)|^2), \qquad \check{\nu}(\zeta) = -\frac{1}{2\pi}\log(1 - |\check{q}(\zeta)|^2). \tag{4.12}$$

Then the L^3-RH problem (4.4) has a unique solution for all sufficiently large τ and this solution satisfies

$$\begin{aligned}(1,1,1)m(\zeta,t,K_1) = (1,1,1) + \frac{2\epsilon}{k_0\sqrt{\tau}}\operatorname{Re}\big(\mathcal{F}_1\beta - \bar{\mathcal{F}}_2\check{\beta}, \mathcal{F}_3\beta - \bar{\mathcal{F}}_3\check{\beta}, \mathcal{F}_2\beta - \bar{\mathcal{F}}_1\check{\beta}\big) \\ + O\big(\epsilon\tau^{-\frac{1+\alpha}{2}}\big), \qquad \tau \to \infty, \quad \zeta \in \mathcal{I},\end{aligned} \tag{4.13}$$

where the error term is uniform with respect to $\zeta \in \mathcal{I}$, the functions $\mathcal{F}_j = \mathcal{F}_j(\zeta)$, $j = 1,2,3$, are defined by

$$\begin{aligned}&\mathcal{F}_1(\zeta) = \frac{1 - k_0^2\omega}{(i + k_0)(1 - k_0K_1)}, \qquad \mathcal{F}_2(\zeta) = \frac{1 - k_0^2\omega}{(i - k_0)(1 + k_0K_1)}, \\ &\mathcal{F}_3(\zeta) = -\frac{i(1 - k_0^2\omega)}{1 + k_0^2\omega},\end{aligned} \tag{4.14}$$

and the functions $\beta = \beta(\zeta,t)$ and $\check{\beta} = \check{\beta}(\zeta,t)$ are defined by

$$\beta(\zeta,t) = \sqrt{\nu(\zeta)}e^{i\left(\frac{\pi}{4} - \arg q(\zeta) + \arg\Gamma(i\nu(\zeta))\right)}e^{-t\phi(\zeta,0)}t^{-i\nu(\zeta)}, \tag{4.15a}$$

$$\check{\beta}(\zeta,t) = \sqrt{\check{\nu}(\zeta)}e^{i\left(\frac{\pi}{4} - \arg\check{q}(\zeta) + \arg\Gamma(i\check{\nu}(\zeta))\right)}e^{-t\phi(\zeta,0)}t^{-i\check{\nu}(\zeta)}. \tag{4.15b}$$

Proof. Since $\det v = 1$ and we are considering an L^3-RH problem for a 3×3-matrix valued function, uniqueness follows from Lemma A.1.

Let m^X be the solution of Theorem B.1 and let

$$\begin{aligned} D(\zeta,t) &= \operatorname{diag}\big(e^{-\frac{t\phi(\zeta,0)}{2}} t^{-\frac{i\nu(\zeta)}{2}}, e^{\frac{t\phi(\zeta,0)}{2}} t^{\frac{i\nu(\zeta)}{2}}, 1\big), \\ \check{D}(\zeta,t) &= \operatorname{diag}\big(e^{-\frac{t\phi(\zeta,0)}{2}} t^{-\frac{i\check{\nu}(\zeta)}{2}}, e^{\frac{t\phi(\zeta,0)}{2}} t^{\frac{i\check{\nu}(\zeta)}{2}}, 1\big). \end{aligned}$$

Define $m_0(\zeta,t,k)$ in neighborhoods of $k=k_0$ and $k=-k_0$ by

$$m_0(\zeta,t,k) = \begin{cases} \mathcal{C}D(\zeta,t) m^X\Big(q(\zeta), -\frac{\sqrt{\tau}}{\epsilon}(k-k_0)\Big) D(\zeta,t)^{-1}\mathcal{C}, & |k-k_0|\le\epsilon, \\ \mathcal{C}\check{D}(\zeta,t)^{-1}\overline{m^X\Big(\check{q}(\zeta), \frac{\sqrt{\tau}}{\epsilon}\overline{(k+k_0)}\Big)}\check{D}(\zeta,t)\mathcal{C}, & |k+k_0|\le\epsilon, \end{cases}$$

and extend it to all of $\mathcal{V}$ in such a way that m_0 obeys the symmetries (2.14a) and (2.14b).

Lemma A.4 implies that m satisfies the L^3-RH problem (4.4) if and only if the function $\hat{m}(\zeta,t,k)$ defined by

$$\hat{m}(\zeta,t,k) = \begin{cases} m(\zeta,t,k)m_0(\zeta,t,k)^{-1}, & k\in\mathcal{V}, \\ m(\zeta,t,k), & \text{otherwise}, \end{cases}$$

satisfies the L^3-RH problem

$$\begin{cases} \hat{m}(\zeta,t,\cdot)\in I+\dot{E}^3(\hat{\mathbb{C}}\setminus\hat{\Gamma}), \\ \hat{m}_+(\zeta,t,k) = \hat{m}_-(\zeta,t,k)\hat{v}(x,t,k) \quad \text{for a.e. } k\in\hat{\Gamma}, \end{cases} \tag{4.16}$$

where the jump matrix $\hat{v}$ is given by

$$\hat{v}(\zeta,t,k) = \begin{cases} m_{0-}(\zeta,t,k)v(\zeta,t,k)m_{0+}(\zeta,t,k)^{-1}, & k\in\mathcal{V}, \\ m_0(\zeta,t,k)^{-1}, & k\in\partial\mathcal{V}, \\ v(\zeta,t,k), & \text{otherwise}. \end{cases}$$

By construction, $\hat{w} = \hat{v} - I$ satisfies the symmetries (2.14a) and (2.14b).

Claim 1. The function $\hat{w} = \hat{v} - I$ satisfies

$$\hat{w}(\zeta,t,k) = O\big(\tau^{-\frac{\alpha}{2}} e^{-\frac{\tau}{24\epsilon^2}|k\mp k_0|^2}\big), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \quad k\in\pm k_0 + X^\epsilon, \tag{4.17}$$

where the error term is uniform with respect to (ζ,k) in the given ranges.

Proof of Claim 1. We first assume $k\in k_0+X^\epsilon$. Then

$$\begin{aligned} \hat{w}(\zeta,t,k) &= m_{0-}(\zeta,t,k)v(\zeta,t,k)m_{0+}(\zeta,t,k)^{-1} - I \\ &= m_{0-}(\zeta,t,k)u(\zeta,t,k)m_{0+}(\zeta,t,k)^{-1}, \end{aligned}$$

where

$$u(\zeta,t,k) := v(\zeta,t,k) - \mathcal{C}D(\zeta,t)v^X\bigg(q(\zeta), -\frac{\sqrt{\tau}}{\epsilon}(k-k_0)\bigg)D(\zeta,t)^{-1}\mathcal{C}.$$

The functions $m_{0+}(\zeta,t,k)$ and $m_{0-}(\zeta,t,k)$ are uniformly bounded for $t>0$, $\zeta\in\mathcal{I}$, and $k\in k_0+X^\epsilon$. Therefore, it is enough to prove that

$$u(\zeta,t,k) = O\big(\tau^{-\frac{\alpha}{2}} e^{-\frac{\tau}{24\epsilon^2}|k-k_0|^2}\big), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \quad k\in k_0+X^\epsilon, \tag{4.18}$$

uniformly with respect to (ζ,k). Introducing the function u_0 by

$$u_0(\zeta,t,z) = \mathcal{C}u\bigg(\zeta,t,k_0 - \frac{\epsilon z}{\rho}\bigg)\mathcal{C} = v_0(\zeta,t,z) - D(\zeta,t)v^X(q(\zeta),\sqrt{t}z)D(\zeta,t)^{-1},$$

we can rewrite the condition (4.18) as follows:

$$u_0(\zeta,t,z) = O\big(\tau^{-\frac{\alpha}{2}} e^{-\frac{t|z|^2}{24}}\big), \qquad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in X^\rho, \tag{4.19}$$

uniformly with respect to (ζ, z) in the given ranges. Using that

$$D(\zeta,t) v^X\big(q(\zeta), \sqrt{t} z\big) D(\zeta,t)^{-1} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ q(\zeta) z^{-2i\nu(\zeta)} e^{\frac{itz^2}{2}} e^{t\phi(\zeta,0)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_1, \\ \begin{pmatrix} 1 & -\frac{\overline{q(\zeta)}}{1-|q(\zeta)|^2} z^{2i\nu(\zeta)} e^{-\frac{itz^2}{2}} e^{-t\phi(\zeta,0)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ -\frac{q(\zeta)}{1-|q(\zeta)|^2} z^{-2i\nu(\zeta)} e^{\frac{itz^2}{2}} e^{t\phi(\zeta,0)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_3, \\ \begin{pmatrix} 1 & \overline{q(\zeta)} z^{2i\nu(\zeta)} e^{-\frac{itz^2}{2}} e^{-t\phi(\zeta,0)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_4, \end{cases}$$

equation (4.19) follows from the assumptions (4.8)-(4.11). Indeed, we will give the details of the proof of (4.19) in the case of $z \in X_1^\rho$; the other cases are similar.

Let $z \in X_1^\rho$. In this case only the (21) entry of $u_0(\zeta,t,z)$ is nonzero and using that $\arg z = \frac{\pi}{4}$ and $\sup_{\zeta\in\mathcal{I}} |q(\zeta)| < 1$, we find

$$\begin{aligned} |(u_0(\zeta,t,z))_{21}| &= \big|R_1(\zeta,t,z) z^{-2i\nu(\zeta)} e^{t\phi(\zeta,z)} - q(\zeta) z^{-2i\nu(\zeta)} e^{\frac{itz^2}{2}} e^{t\phi(\zeta,0)}\big| \\ &= |z^{-2i\nu(\zeta)}| \big|R_1(\zeta,t,z) e^{t\hat{\phi}(\zeta,z)} - q(\zeta)\big| |e^{t\phi(\zeta,0)}| e^{-\frac{t|z|^2}{2}} \\ &\le e^{\frac{\pi\nu(\zeta)}{2}} \Big(\big|R_1(\zeta,t,z) - q(\zeta)\big| e^{t \operatorname{Re}\hat{\phi}(\zeta,z)} + |q(\zeta)| \big|e^{t\hat{\phi}(\zeta,z)} - 1\big|\Big) e^{-\frac{t|z|^2}{2}}, \\ &\qquad \zeta \in \mathcal{I}, \quad t > 0, \quad z \in X_1^\rho, \end{aligned} \tag{4.20}$$

where $\hat{\phi}(\zeta,z) = \phi(\zeta,z) - \phi(\zeta,0) - \frac{iz^2}{2}$. The simple estimate

$$|e^w - 1| = \left|\int_0^1 w e^{sw} ds\right| \le |w| \max_{s\in[0,1]} e^{s\operatorname{Re} w}, \qquad w \in \mathbb{C},$$

yields the inequality

$$|e^w - 1| \le |w| \max(1, e^{\operatorname{Re} w}), \qquad w \in \mathbb{C}. \tag{4.21}$$

On the other hand, by (4.9) and (4.10a),

$$\operatorname{Re}\hat{\phi}(\zeta,z) = \operatorname{Re}\phi(\zeta,z) + \frac{|z|^2}{2} \le \frac{|z|^2}{4}, \qquad \zeta \in \mathcal{I}, \quad z \in X_1^\rho. \tag{4.22}$$

Using (4.21), (4.22), and the fact that $\sup_{\zeta\in\mathcal{I}} |q(\zeta)| < 1$ in (4.20), we find

$$\begin{aligned} |(u_0(\zeta,t,z))_{21}| &\le e^{\frac{\pi\nu(\zeta)}{2}} \Big(\big|R_1(\zeta,t,z) - q(\zeta)\big| + t\big|\hat{\phi}(\zeta,z)\big|\Big) e^{-\frac{t|z|^2}{4}}, \\ &\qquad \zeta \in \mathcal{I}, \quad t > 0, \quad z \in X_1^\rho. \end{aligned}$$

By (4.10c), (4.11), and the fact that $\sup_{\zeta\in\mathcal{I}}|\nu(\zeta)|<\infty$, the right-hand side is of order

$$\begin{aligned}O\bigg(\Big(\frac{L|z|^\alpha}{\rho^\alpha}+\frac{tC|z|^3}{\rho}\Big)e^{-\frac{t|z|^2}{4}}\bigg)&=O\bigg(\Big(\frac{(t|z|^2)^{\alpha/2}}{\tau^{\alpha/2}}+\frac{(t|z|^2)^{3/2}}{\tau^{1/2}}\Big)e^{-\frac{t|z|^2}{12}}\bigg)\\ &=O\bigg(\Big(\frac{1}{\tau^{\alpha/2}}+\frac{1}{\tau^{1/2}}\Big)e^{-\frac{t|z|^2}{24}}\bigg),\qquad \tau\to\infty,\quad \zeta\in\mathcal{I},\quad z\in X_1^\rho,\end{aligned}\tag{4.23}$$

uniformly with respect to (ζ,z) in the given ranges. This proves (4.19) in the case of $z\in X_1^\rho$.

Now let $k\in -k_0+X^\epsilon$. Then

$$\hat{w}(\zeta,t,k)=m_{0-}(\zeta,t,k)v(\zeta,t,k)m_{0+}(\zeta,t,k)^{-1}-I=m_{0-}(\zeta,t,k)u(\zeta,t,k)m_{0+}(\zeta,t,k)^{-1},$$

where

$$u(\zeta,t,k):=v(\zeta,t,k)-\mathcal{C}\overline{\check{D}(\zeta,t)v^X\bigg(\check{q}(\zeta),\frac{\sqrt{\tau}}{\epsilon}\overline{(k+k_0)}\bigg)\check{D}(\zeta,t)^{-1}}\mathcal{C}.$$

The functions $m_{0+}(\zeta,t,k)$ and $m_{0-}(\zeta,t,k)$ are uniformly bounded for $t>0$, $\zeta\in\mathcal{I}$, and $k\in -k_0+X^\epsilon$. Therefore, it is enough to prove that

$$u(\zeta,t,k)=O\big(\tau^{-\frac{\alpha}{2}}e^{-\frac{\tau}{8\epsilon^2}|k+k_0|^2}\big),\qquad \tau\to\infty,\quad \zeta\in\mathcal{I},\quad k\in -k_0+X^\epsilon,\tag{4.24}$$

uniformly with respect to (ζ,k). Introducing the function u_0 by

$$u_0(\zeta,t,z)=\mathcal{C}\overline{u\bigg(\zeta,t,\overline{-k_0+\frac{\epsilon z}{\rho}}\bigg)}\mathcal{C}=\check{v}_0(\zeta,t,z)-\check{D}(\zeta,t)v^X(\check{q}(\zeta),\sqrt{t}z)\check{D}(\zeta,t)^{-1},$$

we can rewrite the condition (4.24) as follows:

$$u_0(\zeta,t,z)=O\big(\tau^{-\frac{\alpha}{2}}e^{-\frac{t|z|^2}{8}}\big),\qquad \tau\to\infty,\quad \zeta\in\mathcal{I},\quad z\in X^\rho,$$

uniformly with respect to (ζ,z) in the given ranges. The rest of the proof is as in the case of $k\in k_0+X^\epsilon$. □

Claim 2. We have

$$\|\hat{w}(\zeta,t,\cdot)\|_{\dot{L}^3(\hat{\Gamma})}=O(\epsilon^{\frac{1}{3}}\tau^{-\frac{\alpha}{2}}),\qquad \tau\to\infty,\quad \zeta\in\mathcal{I},\tag{4.25a}$$

$$\|\hat{w}(\zeta,t,\cdot)\|_{L^\infty(\hat{\Gamma})}=O(\tau^{-\frac{\alpha}{2}}),\qquad \tau\to\infty,\quad \zeta\in\mathcal{I}.\tag{4.25b}$$

Moreover, for any $p\in[1,\infty)$,

$$\|\hat{w}(\zeta,t,\cdot)\|_{\dot{L}^p(\pm k_0+X^\epsilon)}=O(\epsilon^{\frac{1}{p}}\tau^{-\frac{1}{2p}-\frac{\alpha}{2}}),\qquad \tau\to\infty,\quad \zeta\in\mathcal{I}.\tag{4.26}$$

where the error terms are uniform with respect to ζ.

Proof of Claim 2. In view of the symmetries (2.14a) and (2.14b),

$$\begin{aligned}\|\hat{w}(\zeta,t,\cdot)\|_{\dot{L}^3(\hat{\Gamma})}=O\Big(&\|\hat{w}(\zeta,t,\cdot)\|_{\dot{L}^3(\Gamma')}+\|m_0(\zeta,t,\cdot)^{-1}-I\|_{L^3(|k-k_0|=\epsilon)}\\ &+\|m_0(\zeta,t,\cdot)^{-1}-I\|_{L^3(|k+k_0|=\epsilon)}+\|\hat{w}(\zeta,t,\cdot)\|_{L^3(k_0+X^\epsilon)}\\ &+\|\hat{w}(\zeta,t,\cdot)\|_{L^3(-k_0+X^\epsilon)}\Big).\end{aligned}\tag{4.27}$$

On Γ', the matrix $\hat{w}$ is given by either $v-I$ or $m_0(v-I)m_0^{-1}$. Hence $\|\hat{w}(\zeta,t,\cdot)\|_{\dot{L}^3(\Gamma')}=O(\epsilon^{1/3}\tau^{-1})$ by the assumption (4.6a). Moreover, by (B.3), $m^X(q,z)=I+O\big(\frac{1}{z}\big)$ as

$z \to \infty$ uniformly with respect to the argument of z and with respect to q in compact subsets of $\mathbb{D}$. Hence, as the entries of $D(\zeta, t)$ have unit modulus,

$$\begin{aligned}
&\|m_0(\zeta,t,k)^{-1} - I\|_{L^p(|k-k_0|=\epsilon)} \\
&= \left\| \mathcal{C} D(\zeta,t)\left[m^X\left(q(\zeta), -\frac{\sqrt{\tau}}{\epsilon}(k-k_0)\right)^{-1} - I\right] D(\zeta,t)^{-1}\mathcal{C}\right\|_{L^p(|k-k_0|=\epsilon)} \\
&= \begin{cases} O(\epsilon^{1/p}\tau^{-1/2}), & p \in [1,\infty), \\ O(\tau^{-1/2}), & p = \infty, \end{cases}
\end{aligned} \tag{4.28}$$

uniformly with respect to $\zeta \in \mathcal{I}$; the third term on the right-hand side of (4.27) satisfies a similar estimate. The last two terms in (4.27) can be estimated using (4.17). This yields (4.25a). The proof of (4.25b) uses the assumption (4.6b) is similar.

In order to prove (4.26), we note that (4.17) implies

$$\begin{aligned}
\|\hat{w}(\zeta,t,\cdot)\|_{L^p(k_0+X^\epsilon)} &= O\left(\tau^{-\frac{\alpha}{2}}\left(\int_{k_0+X^\epsilon} e^{-\frac{p\tau}{24\epsilon^2}|k-k_0|^2}|dk|\right)^{\frac{1}{p}}\right) \\
&= O\left(\tau^{-\frac{\alpha}{2}}\left(\int_0^\epsilon e^{-\frac{p\tau}{24\epsilon^2}u^2}du\right)^{\frac{1}{p}}\right), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}.
\end{aligned} \tag{4.29}$$

Letting $v = \frac{p\tau}{24\epsilon^2}u^2$ we find

$$\int_0^\epsilon e^{-\frac{p\tau}{24\epsilon^2}u^2}du \le \int_0^\infty e^{-\frac{p\tau}{24\epsilon^2}u^2}du = \frac{\epsilon\sqrt{6}}{\sqrt{p\tau}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv = \frac{\epsilon\sqrt{6\pi}}{\sqrt{p\tau}}. \tag{4.30}$$

Equations (4.29) and (4.30) yield (4.26). □

Let $\hat{\mathcal{C}}$ denote the Cauchy operator associated with $\hat{\Gamma}$:

$$(\hat{\mathcal{C}}f)(z) = \frac{1}{2\pi i}\int_{\hat{\Gamma}} \frac{f(s)}{s-z}ds, \qquad z \in \mathbb{C}\setminus\hat{\Gamma}.$$

Claim 3. There exists a $T > 0$ such that $I - \hat{\mathcal{C}}_{\hat{w}(\zeta,t,\cdot)} \in \mathcal{B}(\dot{L}^3(\hat{\Gamma}))$ is invertible for all $(\zeta,t) \in \mathcal{I}\times(0,\infty)$ with $\tau > T$.

Proof of Claim 3. By (A.5) and (4.25b),

$$\|\hat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))} \le C\|\hat{w}\|_{L^\infty(\hat{\Gamma})} = O(\tau^{-\frac{\alpha}{2}}), \qquad \tau\to\infty. \tag{4.31}$$

This proves the claim. □

In view of Claim 3, we may define the 3×3-matrix valued function $\hat{\mu}(\zeta,t,z)$ whenever $\tau > T$ by

$$\hat{\mu} = I + (I - \hat{\mathcal{C}}_{\hat{w}})^{-1}\hat{\mathcal{C}}_{\hat{w}}I \in I + \dot{L}^3(\hat{\Gamma}). \tag{4.32}$$

Claim 4. The function $\hat{\mu}(\zeta,t,k)$ satisfies

$$\|\hat{\mu}(\zeta,t,\cdot) - I\|_{\dot{L}^3(\hat{\Gamma})} = O\big(\epsilon^{\frac{1}{3}}\tau^{-\frac{\alpha}{2}}\big), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \tag{4.33}$$

where the error term is uniform with respect to $\zeta \in \mathcal{I}$.

Proof of Claim 4. The Neumann series

$$(I - \hat{\mathcal{C}}_{\hat{w}})^{-1} = \sum_{j=0}^\infty \hat{\mathcal{C}}_{\hat{w}}^j \tag{4.34}$$

implies that

$$\|(I-\hat{\mathcal{C}}_{\hat{w}})^{-1}\|_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))} \le \sum_{j=0}^{\infty} \|\hat{\mathcal{C}}_{\hat{w}}\|^j_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))} = \frac{1}{1-\|\hat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))}}.$$

Now (4.3) and the Sokhotski-Plemelj formula $\mathcal{C}_- = \frac{1}{2}(-I+\mathcal{S}_\Gamma)$ show that $\sup_{\zeta\in\mathcal{I}} \|\hat{\mathcal{C}}_-\|_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))} < \infty$. Thus,

$$\begin{aligned}\|\hat{\mu}-I\|_{\dot{L}^3(\hat{\Gamma})} &= \|(I-\hat{\mathcal{C}}_{\hat{w}})^{-1}\hat{\mathcal{C}}_{\hat{w}}I\|_{\dot{L}^3(\hat{\Gamma})} \le \|(I-\hat{\mathcal{C}}_{\hat{w}})^{-1}\|_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))}\|\hat{\mathcal{C}}_-(\hat{w})\|_{\dot{L}^3(\hat{\Gamma})} \\ &\le \frac{C\|\hat{w}\|_{\dot{L}^3(\hat{\Gamma})}}{1-\|\hat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(\dot{L}^3(\hat{\Gamma}))}}.\end{aligned}$$

In view of (4.25a) and (4.31), this gives (4.33). ☐

Claim 5. There exists a unique solution $\hat{m} \in I + \dot{E}^3(\hat{\mathbb{C}} \setminus \hat{\Gamma})$ of the L^3-RH problem (4.16) whenever $\tau > T$. This solution is given by

$$\hat{m}(\zeta,t,k) = I + \hat{\mathcal{C}}_{\hat{w}}\hat{\mu} = I + \frac{1}{2\pi i}\int_{\hat{\Gamma}} \hat{\mu}(\zeta,t,s)\hat{w}(\zeta,t,s)\frac{ds}{s-k}. \tag{4.35}$$

Proof of Claim 5. Uniqueness follows from Lemma A.1 since $\det \hat{v} = 1$. Moreover, equation (4.32) implies that $\hat{\mu} - I = \hat{\mathcal{C}}_{\hat{w}}\hat{\mu}$. Hence, by Lemma A.2, $\hat{m} = I + \hat{\mathcal{C}}(\hat{\mu}\hat{w})$ satisfies the L^3-RH problem (4.16). ☐

Let $C(\zeta)$ denote the union of the two circles $|k-k_0| = \epsilon$ and $|k+k_0| = \epsilon$ oriented counterclockwise. Let $C(\zeta)^{-1}$ denote the image of $C(\zeta)$ under the map $k \mapsto k^{-1}$. The symmetry properties of v imply that $\mathcal{A}\hat{m}(\zeta,t,\omega k)\mathcal{A}^{-1} \in I + \dot{E}^3(\hat{\mathbb{C}} \setminus \hat{\Gamma})$ and $\hat{m}(\zeta,t,k)$ both satisfy the L^3-RH problem (4.16); by uniqueness they are equal, i.e.,

$$\hat{m}(\zeta,t,k) = \mathcal{A}\hat{m}(\zeta,t,\omega k)\mathcal{A}^{-1}, \qquad k \in \hat{\mathbb{C}} \setminus \hat{\Gamma}.$$

Using this symmetry in (4.35), we obtain

$$\begin{aligned} m(\zeta,t,K_1) = \hat{m}(\zeta,t,K_1) = I &+ \frac{1}{2\pi i}\sum_{n=0}^{2} \mathcal{A}^{-n}\big[F_n(\zeta,t) + G_n(\zeta,t)\big]\mathcal{A}^n \\ &+ \frac{1}{2\pi i}\int_{\Gamma} \hat{\mu}(\zeta,t,k)\hat{w}(\zeta,t,k)\frac{dk}{k-K_1}. \end{aligned} \tag{4.36}$$

where

$$F_n(\zeta,t) = \int_{C(\zeta)} \frac{\hat{\mu}(\zeta,t,k)\hat{w}(\zeta,t,k)dk}{k-\omega^{-n}K_1}, \qquad G_n(\zeta,t) = \int_{C(\zeta)^{-1}} \frac{\hat{\mu}(\zeta,t,k)\hat{w}(\zeta,t,k)dk}{k-\omega^{-n}K_1}.$$

By (B.3),

$$\begin{aligned} m_0(\zeta,t,k)^{-1} &= \mathcal{C}D(\zeta,t)m^X\bigg(q(\zeta), -\frac{\sqrt{\tau}}{\epsilon}(k-k_0)\bigg)^{-1} D(\zeta,t)^{-1}\mathcal{C} \\ &= I - \frac{B(\zeta,t)}{\sqrt{\tau}(k-k_0)} + O(\tau^{-1}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \quad |k-k_0| = \epsilon, \end{aligned} \tag{4.37a}$$

and

$$\begin{aligned} m_0(\zeta,t,k)^{-1} &= \mathcal{C}\check{D}(\zeta,t)^{-1}\overline{m^X\bigg(\check{q}(\zeta), \frac{\sqrt{\tau}}{\epsilon}\overline{(k+k_0)}\bigg)^{-1}}\check{D}(\zeta,t)\mathcal{C} \\ &= I + \frac{\overline{\check{B}(\zeta,t)}}{\sqrt{\tau}(k+k_0)} + O(\tau^{-1}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \quad |k+k_0| = \epsilon, \end{aligned} \tag{4.37b}$$

where $B(\zeta,t)$ and $\check{B}(\zeta,t)$ are defined by

$$B(\zeta,t) = i\epsilon\begin{pmatrix} 0 & -\beta(\zeta,t) & 0 \\ \overline{\beta(\zeta,t)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \check{B}(\zeta,t) = i\epsilon\begin{pmatrix} 0 & -\check{\beta}(\zeta,t) & 0 \\ \overline{\check{\beta}(\zeta,t)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$\beta(\zeta,t) = \beta^X(q(\zeta))e^{-t\phi(\zeta,0)}t^{-i\nu(\zeta)}, \qquad \check{\beta}(\zeta,t) = \beta^X(\check{q}(\zeta))e^{-t\phi(\zeta,0)}t^{-i\check{\nu}(\zeta)}.$$

Using (4.28), (4.33), and (4.37) we find

$$\begin{aligned}
F_n(\zeta,t) &= \int_{C(\zeta)} \frac{\hat{\mu}(\zeta,t,k)(m_0(\zeta,t,k)^{-1}-I)dk}{k-\omega^{-n}K_1} \\
&= \int_{C(\zeta)} \frac{(m_0(\zeta,t,k)^{-1}-I)dk}{k-\omega^{-n}K_1} + \int_{C(\zeta)} \frac{(\hat{\mu}(\zeta,t,k)-I)(m_0(\zeta,t,k)^{-1}-I)dk}{k-\omega^{-n}K_1} \\
&= -\int_{|k-k_0|=\epsilon} \frac{B(\zeta,t)}{\sqrt{\tau}(k-k_0)}\frac{dk}{k-\omega^{-n}K_1} + \int_{|k+k_0|=\epsilon} \frac{\overline{\check{B}(\zeta,t)}}{\sqrt{\tau}(k+k_0)}\frac{dk}{k-\omega^{-n}K_1} \\
&\quad + O(\epsilon\tau^{-1}) + O\big(\|\hat{\mu}-I\|_{\dot{L}^3(\hat{\Gamma})}\|m_0^{-1}-I\|_{\dot{L}^{3/2}(C(\zeta))}\big) \\
&= -\frac{2\pi i}{\sqrt{\tau}}\left(\frac{B(\zeta,t)}{k_0-\omega^{-n}K_1} + \frac{\overline{\check{B}(\zeta,t)}}{k_0+\omega^{-n}K_1}\right) + O(\epsilon\tau^{-\frac{1+\alpha}{2}})
\end{aligned} \tag{4.38}$$

uniformly with respect to $\zeta \in \mathcal{I}$.

In order to compute the contribution from G_n we note that (4.37) implies

$$\begin{aligned}
m_0(\zeta,t,k)^{-1} &= \mathcal{B}m_0(\zeta,t,k^{-1})^{-1}\mathcal{B} \\
&= \begin{cases} I - \frac{\mathcal{B}B(\zeta,t)\mathcal{B}}{\sqrt{\tau}(k^{-1}-k_0)} + O(\tau^{-1}), & |k^{-1}-k_0| = \epsilon, \\ I + \frac{\mathcal{B}\overline{\check{B}(\zeta,t)}\mathcal{B}}{\sqrt{\tau}(k^{-1}+k_0)} + O(\tau^{-1}), & |k^{-1}+k_0| = \epsilon, \end{cases} \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}.
\end{aligned}$$

Hence, proceeding as in (4.38), we find

$$\begin{aligned}
G_n(\zeta,t) &= \int_{C(\zeta)^{-1}} \frac{\hat{\mu}(\zeta,t,k)(m_0(\zeta,t,k)^{-1}-I)dk}{k-\omega^{-n}K_1} \\
&= \int_{C(\zeta)^{-1}} \frac{(m_0(\zeta,t,k)^{-1}-I)dk}{k-\omega^{-n}K_1} + \int_{C(\zeta)^{-1}} \frac{(\hat{\mu}(\zeta,t,k)-I)(m_0(\zeta,t,k)^{-1}-I)dk}{k-\omega^{-n}K_1} \\
&= -\int_{|k^{-1}-k_0|=\epsilon} \frac{\mathcal{B}B(\zeta,t)\mathcal{B}}{\sqrt{\tau}(k^{-1}-k_0)}\frac{dk}{k-\omega^{-n}K_1} + \int_{|k^{-1}+k_0|=\epsilon} \frac{\mathcal{B}\overline{\check{B}(\zeta,t)}\mathcal{B}}{\sqrt{\tau}(k^{-1}+k_0)}\frac{dk}{k-\omega^{-n}K_1} \\
&\quad + O(\epsilon\tau^{-1}) + O\big(\|\hat{\mu}-I\|_{\dot{L}^3(\hat{\Gamma})}\|m_0^{-1}-I\|_{\dot{L}^{3/2}(C(\zeta)^{-1})}\big) \\
&= \frac{2\pi i}{\sqrt{\tau}}\left(\frac{\mathcal{B}B(\zeta,t)\mathcal{B}}{k_0(1-k_0\omega^{-n}K_1)} + \frac{\mathcal{B}\overline{\check{B}(\zeta,t)}\mathcal{B}}{k_0(1+k_0\omega^{-n}K_1)}\right) + O(\epsilon\tau^{-\frac{1+\alpha}{2}})
\end{aligned} \tag{4.39}$$

uniformly with respect to $\zeta \in \mathcal{I}$.

On the other hand, using (4.2),

$$\begin{aligned}
\left|\int_\Gamma \frac{\hat{\mu}(\zeta,t,k)\hat{w}(\zeta,t,k)dk}{k-K_1}\right| &= \left|\int_\Gamma \frac{(\hat{\mu}(\zeta,t,k)-I)\hat{w}(\zeta,t,k)dk}{k-K_1} + \int_\Gamma \frac{\hat{w}(\zeta,t,k)dk}{k-K_1}\right| \\
&\le C\|\hat{\mu}-I\|_{\dot{L}^3(\Gamma)}\|\hat{w}\|_{\dot{L}^{\frac{3}{2}}(\Gamma)} + C\|\hat{w}\|_{\dot{L}^1(\Gamma)}.
\end{aligned}$$

The $\dot{L}^1$-norm of $\hat{w}$ is $O(\epsilon\tau^{-1})$ on Γ' by (4.6a) and is $O(\epsilon\tau^{-\frac{1+\alpha}{2}})$ on $\{\pm k_0 + X^\epsilon\}$ by (4.26). Hence $\|\hat{w}\|_{\dot{L}^1(\Gamma)} = O(\epsilon\tau^{-\frac{1+\alpha}{2}})$. Similarly, $\|\hat{w}\|_{\dot{L}^{\frac{3}{2}}(\Gamma)} = O(\epsilon^{\frac{2}{3}}\tau^{-1} + \epsilon^{\frac{2}{3}}\tau^{-\frac{1}{3}-\frac{\alpha}{2}})$ by

(4.6a) and (4.26). Since $\|\hat{\mu} - I\|_{\dot{L}^3(\Gamma)} = O(\epsilon^{1/3}\tau^{-\frac{\alpha}{2}})$ by (4.33) and $1/3 \le \alpha < 1$, we infer that

$$\left|\int_\Gamma \frac{\hat{\mu}(\zeta,t,k)\hat{w}(\zeta,t,k)dk}{k-K_1}\right| = O(\epsilon\tau^{-\frac{1+\alpha}{2}}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \tag{4.40}$$

uniformly with respect to $\zeta \in \mathcal{I}$. Equations (4.36), (4.38), (4.39), and (4.40) yield

$$\begin{aligned} m(\zeta,t,K_1) = I - \frac{1}{\sqrt{\tau}}\sum_{n=0}^{2}\mathcal{A}^{-n}\bigg(&\frac{B(\zeta,t)}{k_0-\omega^{-n}K_1} + \frac{\overline{\check{B}(\zeta,t)}}{k_0+\omega^{-n}K_1} \\ &- \frac{\mathcal{B}B(\zeta,t)\mathcal{B}}{k_0(1-k_0\omega^{-n}K_1)} - \frac{\mathcal{B}\overline{\check{B}(\zeta,t)}\mathcal{B}}{k_0(1+k_0\omega^{-n}K_1)}\bigg)\mathcal{A}^n + O(\epsilon\tau^{-\frac{1+\alpha}{2}}). \end{aligned} \tag{4.41}$$

Recalling that $K_1 = e^{\frac{i\pi}{6}}$, a tedious but straightforward computation gives (4.13). □

5. Long-time asymptotics in the similarity sector

Theorem 5.1 gives exact formulas for the leading order asymptotics of the solution of the DP equation on the half-line in the asymptotic region

$$0 < c < \xi < 3, \qquad (3-\xi)^{\frac{3}{2}}t \to \infty,$$

where $\xi = x/t$.

Theorem 5.1. *Let u_0 and $\{g_j\}_0^2$ be functions in the Schwartz class $\mathcal{S}(\mathbb{R}_+)$ that satisfy the assumptions (2.2). Suppose there exists a unique solution $u(x,t)$ of equation (1.1) with $\kappa = 1$ in the half-line domain $\Omega = \{x \ge 0, t \ge 0\}$ such that*

- *u is a smooth function of $(x,t) \in \Omega$,*
- *u satisfies the initial and boundary conditions (2.1),*
- *$u(\cdot,t) \in \mathcal{S}(\mathbb{R}_+)$ for each $t \ge 0$.*

Let $q(x,t) = (u(x,t) - u_{xx}(x,t) + 1)^{\frac{1}{3}}$. Define $\tilde{\Phi}_n(x,t,k)$ for (x,t) in the set

$$\{x \ge 0, t = 0\} \cup \{x = 0, t \ge 0\}$$

by the linear integral equations (2.11b). Define $r(k)$ for $k \in (\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2})$ by

$$r(k) = (\tilde{\Phi}_{18}(0,0,k)^{-1}\tilde{\Phi}_7(0,0,k))_{21}e^{i(k-k^{-1})\int_0^\infty (q(x,0)-1)dx}.$$

Suppose the set $\{k_j\}$ defined in Section 2 is empty and that

$$\sup_{\frac{\sqrt{5}-1}{2}<k<\frac{\sqrt{5}+1}{2}} |r(k)| < 1.$$

Then, for any $\alpha \in [\frac{1}{3}, 1)$ and $c > 0$, the following asymptotic formulas are valid:

$$q(x,t) = 1 + \frac{b_1(\xi)}{\sqrt{t}}\cos\big(b_2(\xi)t - \nu(\xi)\log(t) + b_3(\xi)\big) + O\big((3-\xi)^{-\frac{1+3\alpha}{4}}t^{-\frac{1+\alpha}{2}}\big),$$
$$(3-\xi)^{\frac{3}{2}}t \to \infty, \quad c \le \xi < 3, \tag{5.1}$$

$$u(x,t) = \frac{3b_1(\xi)}{(1+4\tilde{k}_0^2(\xi))\sqrt{t}}\cos\big(b_2(\xi)t - \nu(\xi)\log t + b_3(\xi)\big) + O\big((3-\xi)^{-\frac{1+3\alpha}{4}}t^{-\frac{1+\alpha}{2}}\big),$$
$$(3-\xi)^{\frac{3}{2}}t \to \infty, \quad c \le \xi < 3, \tag{5.2}$$

where the error terms are uniform with respect to ξ in the given ranges and the functions $\{b_j(\xi)\}_1^3$, $\nu(\xi)$, and $\tilde{k}_0(\xi)$ are defined by

$$b_1(\xi) = \frac{1-k_0^2(\xi)}{1+k_0^2(\xi)}\sqrt{\frac{(3+4\tilde{k}_0^2(\xi))(1+4\tilde{k}_0^2(\xi))\nu(\xi)}{3\tilde{k}_0(\xi)-4\tilde{k}_0^3(\xi)}}, \qquad b_2(\xi) = \frac{48\tilde{k}_0^3(\xi)}{(1+4\tilde{k}_0^2(\xi))^2},$$

$$\begin{aligned} b_3(\xi) = {} & \frac{\pi}{4} - \chi_0(\xi) + \nu(\xi)\log\left(\frac{(4\tilde{k}_0^2(\xi)+1)^2(4\tilde{k}_0^2(\xi)+3)}{576\tilde{k}_0^3(\xi)(3-4\tilde{k}_0^2(\xi))}\right) + \arg\Gamma(i\nu(\xi)) - \arg r(k_0(\xi)) \\ & - \arctan\left(\sqrt{3}\frac{1+k_0^2(\xi)}{1-k_0^2(\xi)}\right) + \frac{3\tilde{k}_0(\xi)}{\pi}\int_{k_0(\xi)}^{\frac{1}{k_0(\xi)}} \log(1-|r(s)|^2)\frac{1+s^4}{1+s^6}ds, \end{aligned}$$

$$\nu(\xi) = -\frac{1}{2\pi}\log(1-|r(k_0(\xi))|^2), \qquad \tilde{k}_0(\xi) = \sqrt{\frac{-2\xi-3+\sqrt{24\xi+9}}{8\xi}},$$

with

$$\begin{aligned} k_0(\xi) = {} & -\tilde{k}_0(\xi) + \sqrt{1+\tilde{k}_0^2(\xi)}, \\ \chi_0(\xi) = {} & -\frac{3}{2\pi}\int_{k_0(\xi)}^{\frac{1}{k_0(\xi)}} \log\left(\frac{1-|r(s)|^2}{1-|r(k_0(\xi))|^2}\right) \\ & \times \frac{(k_0(\xi)-k_0^{11}(\xi))(s^6+s^4)+(k_0^5(\xi)-k_0^7(\xi))(s^{10}+1)}{k_0^{12}(\xi)s^6 - k_0^6(\xi)(s^{12}+1)+s^6}ds. \end{aligned} \tag{5.3}$$

5.1. **Proof of Theorem 5.1.** The proof of Theorem 5.1 will proceed through several steps. The basic idea consists of deforming the contour of the RH problem (3.6) so that the jump matrix is exponentially small everywhere except near a certain set of critical points.

Suppose $\zeta = y/t \in (0,3)$. In order to find the critical points, we note that, for fixed ζ, the function $\Phi(\zeta,k)$ defined in (3.8) can be written as $\Phi(\zeta,k) = F(\tilde{k}(k))$ where

$$F(\tilde{k}) = 2i\zeta\tilde{k} - \frac{6i\tilde{k}}{1+4\tilde{k}^2}, \qquad \tilde{k}(k) = \frac{1}{2}\left(\frac{1}{k}-k\right).$$

The equation $F'(\tilde{k}) = 0$ has two real solutions $\pm\tilde{k}_0$, where $\tilde{k}_0 = \tilde{k}_0(\zeta)$ is defined by

$$\tilde{k}_0 = \sqrt{\frac{-2\zeta-3+\sqrt{24\zeta+9}}{8\zeta}}.$$

Consequently, there are four real points at which $\partial\Phi/\partial k = 0$; these are given by $\pm k_0, \pm k_0^{-1}$ where $k_0 = k_0(\zeta)$ is defined by

$$k_0 = -\tilde{k}_0 + \sqrt{1+\tilde{k}_0^2}.$$

It follows that for $\zeta \in (0,3)$ the RH problem (3.6) has twelve critical points associated with it (the four points $\pm k_0, \pm k_0^{-1}$ as well as the eight points obtained by multiplying these four points by ω and ω^2). Note that $0 \le \tilde{k}_0 \le 1/2$ and $\frac{\sqrt{5}-1}{2} \le k_0 \le 1$ for $0 \le \zeta \le 3$.

Step 1: Deform contour. By introducing rational approximations if necessary, we may assume that the jump matrices in (3.3) admit appropriate analytic continuations, see [12]. We begin by deforming the contour so that it (a) avoids the points $\{K_j\}_1^6$ and (b) passes through the twelve critical points, see Figures 5 and 6.

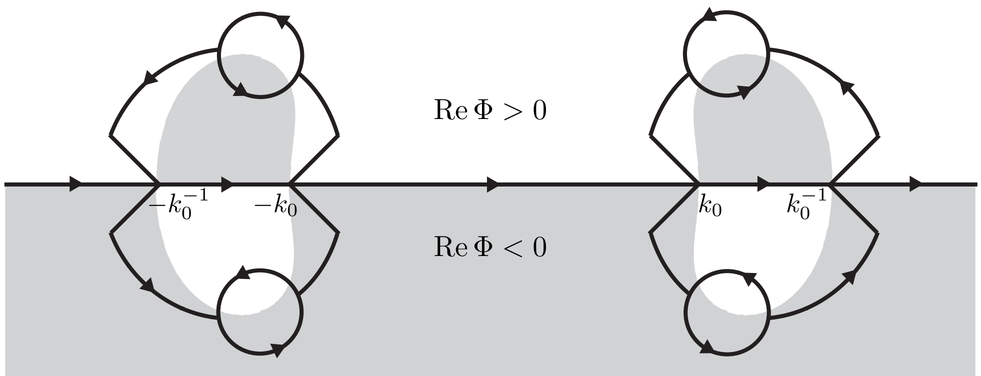

Figure 5. *The jump contour* $\hat{\Gamma}$ *for* k *near* $\mathbb{R}$ *together with the regions where* $\operatorname{Re}\Phi < 0$ *(shaded) and* $\operatorname{Re}\Phi > 0$ *(unshaded).*

For k near K_1, the contour deformation is achieved by letting

$$\hat{M} = \begin{cases} MJ_{19,25}J_{19,26} & \text{in} \quad E_{19}, \\ MJ_{19,26} & \text{in} \quad E_{25}, \\ MJ_{19,26}^{-1}J_{19,25}J_{19,26} & \text{in} \quad E_{26}; \end{cases}$$

note that the function $\hat{M}$ has no jump across $\bar{E}_{19} \cap \bar{E}_{25}$ and $\bar{E}_{19} \cap \bar{E}_{26}$. We use the symmetries (2.14) to extend this definition to $\cup_{n=1}^{18} E_{n+18}$.

For k near k_0 and k_0^{-1}, the contour deformation is achieved by defining $\hat{M} = MJ_{1,7}^{-1}$ for $k \in F_1 \cap E_7$ and $\hat{M} = MJ_{6,18}^{-1}$ for $k \in F_6 \cap E_{18}$, where the sets $\{F_n\}$ are as in Figure 6. We define $\hat{M}$ analogously near the other critical points and set $\hat{M} = M$ otherwise.

Let $\hat{\Gamma}$ denote the contour displayed in Figure 6. Proposition 3.1 and the expressions (3.7) and (3.9) for J show that, except for possible singularities at the points $\{\varkappa_j\}_1^6$, $\hat{M}(y,t,k)$ is a bounded and analytic function of $k \in \hat{\mathbb{C}} \setminus \hat{\Gamma}$. Indeed, the matrix $J_{1,7}$ involves the exponential $e^{-t\Phi}$ which is bounded for $k \in \bar{F}_1 \cap \bar{E}_7$; hence $\hat{M}$ is bounded in F_1. Similarly, $\hat{M}$ is bounded in E_{19} because $\operatorname{Re} z_1 < \operatorname{Re} z_2 < \operatorname{Re} z_3$ in E_{19} and

$$J_{19,25}J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}; \tag{5.4a}$$

$\hat{M}$ is bounded in E_{25} because $\operatorname{Re} z_2 < \operatorname{Re} z_3$ in E_{25} and

$$J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}; \tag{5.4b}$$

and $\hat{M}$ is a bounded in E_{26} because $\operatorname{Re} z_1 < \operatorname{Re} z_3 < \operatorname{Re} z_2$ in E_{26} and

$$J_{19,26}^{-1}J_{19,25}J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4c}$$

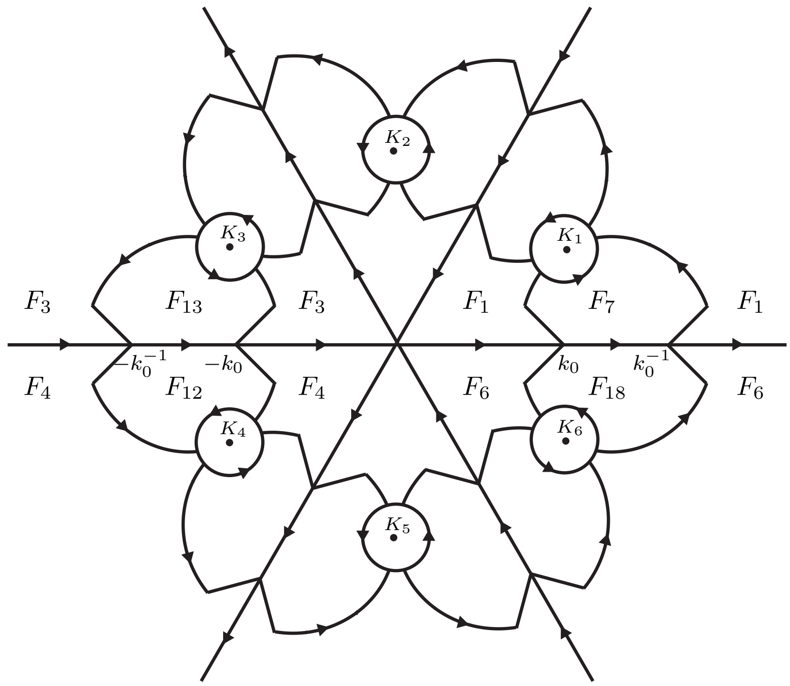

Figure 6. *The jump contour $\hat{\Gamma}$ for $\hat{M}$.*

Using Lemma A.4 we conclude that $\hat{N} = (1, 1, 1)\hat{M}$ satisfies the L^3-RH problem

$$\begin{cases} \hat{N}(y,t,\cdot) \in (1,1,1) + \dot{E}^3(\hat{\mathbb{C}} \setminus \hat{\Gamma}), \\ \hat{N}(y,t,k) = \hat{N}(y,t,k)\hat{J}(y,t,k) \quad \text{for a.e. } k \in \hat{\Gamma}. \end{cases} \tag{5.5}$$

where, by (3.10),

$$\hat{J} = \begin{cases} J_{1,19}J_{19,25}J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, & \bar{E}_1 \cap \bar{E}_{19}, \\ J_{7,25}J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, & \bar{E}_7 \cap \bar{E}_{25}, \\ J_{8,26}J_{19,26}^{-1}J_{19,25}J_{19,26} = e^{y\hat{\mathcal{L}}+t\hat{\mathcal{Z}}} \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \bar{F}_8 \cap \bar{F}_{26}. \end{cases} \tag{5.6}$$

Step 2: Conjugate. On the circles where the E_n's and E_{n+18}'s meet, the jump matrix $\hat{J}(y,t,k)$ has decay as $t \to \infty$. Indeed, the (12), (13), and (23) entries of the matrices in (5.6) involve the exponentials $e^{-t\Phi(\zeta,k)}$, $e^{t\Phi(\zeta,\omega^2 k)}$, and $e^{-t\Phi(\zeta,\omega k)}$, respectively; the decay now follows from the signature table of $\operatorname{Re}\Phi$, see Figure 5.

In order to arrive at a jump matrix with the appropriate decay properties also on the remaining part of the contour, we need to perform a triangular factorization of $\hat{J}$.

Such a factorization can be achieved by conjugating the RH problem as follows. Let

$$\Delta(\zeta,k)=\begin{pmatrix}\delta(\zeta,k)\delta(\zeta,\omega^2k)^{-1} & 0 & 0\\ 0 & \delta(\zeta,k)^{-1}\delta(\zeta,\omega k) & 0\\ 0 & 0 & \delta(\zeta,\omega^2k)\delta(\zeta,\omega k)^{-1}\end{pmatrix},$$

where

$$\begin{aligned}\delta(\zeta,k)=\exp\Bigg\{&\frac{1}{4\pi i}\int_{-\frac{1}{k_0}}^{-k_0}\log(1-|\check{r}(s)|^2)\bigg(\frac{1}{s-k}-\frac{1}{s-\frac{1}{k}}\bigg)ds\\ &+\frac{1}{4\pi i}\int_{k_0}^{\frac{1}{k_0}}\log(1-|r(s)|^2)\bigg(\frac{1}{s-k}-\frac{1}{s-\frac{1}{k}}\bigg)ds\Bigg\},\qquad k\in\mathbb{C}\setminus\mathbb{R}.\end{aligned}\tag{5.7}$$

The identities

$$\delta(\zeta,k)=\frac{1}{\overline{\delta(\zeta,\bar{k})}}=\frac{1}{\delta(\zeta,k^{-1})},\qquad k\in\mathbb{C},$$

imply that $\Delta(\zeta,k)$ obeys the three symmetries in (2.14). The function δ satisfies

$$\delta_+(\zeta,k)=\begin{cases}\delta_-(\zeta,k)(1-|r(k)|^2), & k\in(k_0,k_0^{-1}),\\ \delta_-(\zeta,k)(1-|\check{r}(k)|^2), & k\in(-k_0^{-1},-k_0),\\ \delta_-(\zeta,k), & \text{otherwise},\end{cases}$$

and

$$\delta(\zeta,k)=e^{i\varphi}+O(k^{-1}),\qquad k\to\infty,\quad k\in\mathbb{C},$$

where the constant $\varphi\in\mathbb{R}$ is given by

$$\varphi=\frac{1}{4\pi}\int_{-\frac{1}{k_0}}^{-k_0}\log(1-|\check{r}(s)|^2)\frac{ds}{s}+\frac{1}{4\pi}\int_{k_0}^{\frac{1}{k_0}}\log(1-|r(s)|^2)\frac{ds}{s}.$$

Moreover, the representation

$$\delta(\zeta,k)=\bigg(\frac{(k_0^{-1}-k)(k_0-k^{-1})}{(k_0-k)(k_0^{-1}-k^{-1})}\bigg)^{\frac{i\nu}{2}}\bigg(\frac{(k_0+k)(k_0^{-1}+k^{-1})}{(k_0^{-1}+k)(k_0+k^{-1})}\bigg)^{\frac{i\check{\nu}}{2}}e^{\chi(\zeta,k)},$$

where

$$\begin{aligned}\chi(\zeta,k)=&\frac{1}{4\pi i}\int_{-\frac{1}{k_0}}^{-k_0}\log\bigg(\frac{1-|\check{r}(s)|^2}{1-|\check{r}(-k_0)|^2}\bigg)\bigg(\frac{1}{s-k}-\frac{1}{s-\frac{1}{k}}\bigg)ds\\ &+\frac{1}{4\pi i}\int_{k_0}^{\frac{1}{k_0}}\log\bigg(\frac{1-|r(s)|^2}{1-|r(k_0)|^2}\bigg)\bigg(\frac{1}{s-k}-\frac{1}{s-\frac{1}{k}}\bigg)ds,\end{aligned}$$

shows that

$$\delta(\zeta,\cdot),\delta(\zeta,\cdot)^{-1}\in E^\infty(\mathbb{C}\setminus\mathbb{R}).$$

We conclude that

$$\Delta(\zeta,\cdot),\Delta(\zeta,\cdot)^{-1}\in I+\dot{E}^3(\mathbb{C}\setminus\hat{\Gamma})\cap E^\infty(\mathbb{C}\setminus\hat{\Gamma}).$$

The function $\tilde{M}$ defined by

$$\tilde{M}(y,t,k)=\hat{M}(y,t,k)\Delta(\zeta,k)$$

satisfies the jump condition $\tilde{M}_+ = \tilde{M}_- \tilde{J}$ on $\hat{\Gamma}$ with $\tilde{J} = \Delta_-^{-1} \hat{J} \Delta_+$. We find from (3.7) that

$$\tilde{J} = \begin{cases} b_l^{-1} b_u, & k \in \bar{F}_1 \cap \bar{F}_6, \\ B_u^{-1} B_l, & k \in \bar{F}_7 \cap \bar{F}_{18}, \\ \begin{pmatrix} 1 & \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2} \overline{h(\bar{k})} e^{-t\Phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{F}_1 \cap \bar{F}_7, \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)} h(k) e^{t\Phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{F}_6 \cap \bar{F}_{18}, \\ \check{b}_l^{-1} \check{b}_u, & k \in \bar{F}_3 \cap \bar{F}_4, \\ \check{B}_u^{-1} \check{B}_l, & k \in \bar{F}_{12} \cap \bar{F}_{13}, \\ \begin{pmatrix} 1 & \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2} \overline{\check{h}(\bar{k})} e^{-t\Phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{F}_3 \cap \bar{F}_{12}, \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)} \check{h}(k) e^{t\Phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{F}_4 \cap \bar{F}_{13}. \end{cases}$$

where

$$b_l = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)} (r(k) + h(k)) e^{t\Phi(\zeta,k)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_u = \begin{pmatrix} 1 & \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta_-(\zeta,k)^2} \frac{\overline{r(\bar{k})}}{1 - r(k)\overline{r(\bar{k})}} e^{-t\Phi(\zeta,k)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_l = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_+(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)} \frac{r(k)}{1 - r(k)\overline{r(\bar{k})}} e^{t\Phi(\zeta,k)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b_u = \begin{pmatrix} 1 & -\frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2} (\overline{r(\bar{k})} + \overline{h(\bar{k})}) e^{-t\Phi(\zeta,k)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and $\check{b}_l$, $\check{B}_u$, $\check{B}_l$, $\check{b}_u$ are given by analogous expressions but with r and h replaced with $\check{r}$ and $\check{h}$, respectively.

Step 3: Deform again. Let the sets $\{U_j\}$ and $\{V_j\}$ be as in Figure 7 and define $\mathfrak{m}(y,t,k)$ for k near $\mathbb{R}$ by

$$\mathfrak{m}(y,t,k) = \begin{cases} \tilde{M}(y,t,k)B_l(y,t,k)^{-1}, & k \in U_1 \cap \mathbb{C}_+, \\ \tilde{M}(y,t,k)b_u(y,t,k)^{-1}, & k \in U_3 \cap \mathbb{C}_+, \\ \tilde{M}(y,t,k)b_l(y,t,k)^{-1}, & k \in U_3 \cap \mathbb{C}_-, \\ \tilde{M}(y,t,k)B_u(y,t,k)^{-1}, & k \in U_1 \cap \mathbb{C}_-, \\ \tilde{M}(y,t,k)\check{B}_l(y,t,k)^{-1}, & k \in U_5 \cap \mathbb{C}_+, \\ \tilde{M}(y,t,k)\check{b}_u(y,t,k)^{-1}, & k \in U_7 \cap \mathbb{C}_+, \\ \tilde{M}(y,t,k)\check{b}_l(y,t,k)^{-1}, & k \in U_7 \cap \mathbb{C}_-, \\ \tilde{M}(y,t,k)\check{B}_u(y,t,k)^{-1}, & k \in U_5 \cap \mathbb{C}_-. \end{cases} \tag{5.8}$$

We define $\mathfrak{m}$ analogously near the lines $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$ and set $\mathfrak{m} = \tilde{M}$ elsewhere. Let Γ denote the jump contour for $\mathfrak{m}$; the part of Γ near $\mathbb{R}$ is displayed in Figure 7. The function $\mathfrak{m}$ satisfies the jump condition $\mathfrak{m}_+ = \mathfrak{m}_- v$ on Γ, where

$$v(y,t,k) = \begin{cases} B_l, & k \in \bar{U}_1 \cap \bar{U}_2, \\ b_u, & k \in \bar{U}_3 \cap \bar{V}_1, \\ b_l, & k \in \bar{U}_3 \cap \bar{V}_6, \\ B_u, & k \in \bar{U}_1 \cap \bar{U}_4, \\ b_u \tilde{J}_{1,7} = \begin{pmatrix} 1 & -\frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2}\overline{r(\bar{k})}e^{-t\Phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{U}_2 \cap \bar{U}_3, \\ b_l \tilde{J}_{6,18} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)} r(k) e^{t\Phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{U}_3 \cap \bar{U}_4, \\ \check{B}_l, & k \in \bar{U}_5 \cap \bar{U}_6, \\ \check{b}_u, & k \in \bar{U}_7 \cap \bar{V}_3, \\ \check{b}_l, & k \in \bar{U}_7 \cap \bar{V}_4, \\ \check{B}_u, & k \in \bar{U}_5 \cap \bar{U}_8, \\ \check{b}_u \tilde{J}_{3,12} = \begin{pmatrix} 1 & -\frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2}\overline{\check{r}(\bar{k})}e^{-t\Phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{U}_6 \cap \bar{U}_7, \\ \check{b}_l \tilde{J}_{4,13} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)} \check{r}(k) e^{t\Phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & k \in \bar{U}_7 \cap \bar{U}_8. \end{cases} \tag{5.9}$$

The jump matrix v obeys the symmetries (2.14a) and (2.14b).

Equation (5.5) and Lemma A.4 imply that, for each $(y,t) \in F(\Omega)$, the function $n(y,t,k)$ defined by

$$n(y,t,k) = (1,1,1)\mathfrak{m}(y,t,k), \qquad k \in \hat{\mathbb{C}} \setminus \Gamma \tag{5.10}$$

is a row vector solution of the L^3-RH problem

$$\begin{cases} n(y,t,\cdot) \in I + \dot{E}^3(\hat{\mathbb{C}} \setminus \Gamma), \\ n_+(y,t,k) = n_-(y,t,k)v(y,t,k) \quad \text{for a.e. } k \in \Gamma. \end{cases} \tag{5.11}$$

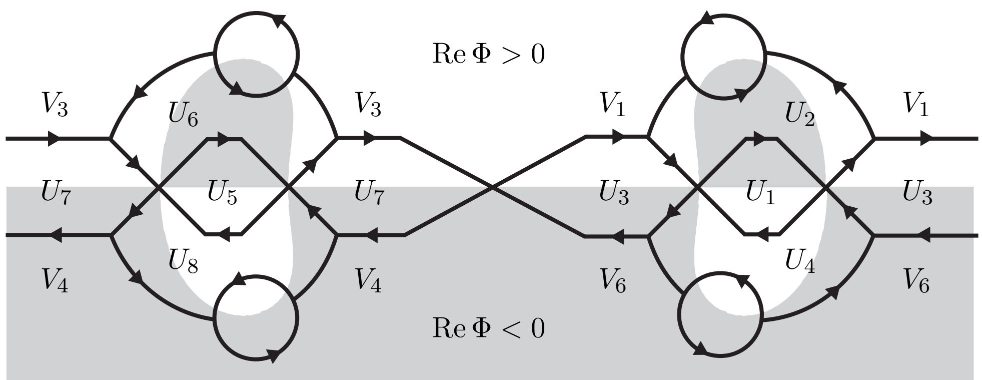

Figure 7. *The jump contour for* $\mathfrak{m}$ *for* k *near* $\mathbb{R}$.

Step 4: Apply Theorem 4.1. Let $0 < \epsilon < 1$, $\alpha \in [\frac{1}{3}, 1)$, and $c \in (0, 3)$. We claim that Theorem 4.1 can be applied to the contour Γ and jump matrix v with

$$\begin{aligned}
&\mathcal{I} = [c, 3), \qquad \rho = \epsilon\sqrt{-iF''(\tilde{k}_0)\tilde{k}'(k_0)^2} = \epsilon\frac{1+k_0^{-2}}{2}\sqrt{\frac{48\tilde{k}_0(3-4\tilde{k}_0^2)}{(4\tilde{k}_0^2+1)^3}},\\
&q(\zeta) = \left(\frac{2}{k_0^{-1}+k_0}\right)^{2i\check{\nu}(\zeta)}\frac{e^{2\chi(\zeta,k_0)}}{\delta(\zeta,\omega k_0)\delta(\zeta,\omega^2 k_0)}r(k_0)\left(\frac{\rho}{\epsilon}(k_0^{-1}-k_0)k_0\right)^{2i\nu(\zeta)},\\
&\check{q}(\zeta) = \left(\frac{2}{k_0^{-1}+k_0}\right)^{2i\nu(\zeta)}\frac{\delta(\zeta,-\omega k_0)\delta(\zeta,-\omega^2 k_0)}{e^{2\chi(\zeta,-k_0)}}\overline{\check{r}(-k_0)}\left(\frac{\rho}{\epsilon}(k_0^{-1}-k_0)k_0\right)^{2i\check{\nu}(\zeta)},\\
&\nu(\zeta) = -\frac{1}{2\pi}\log(1-|r(k_0)|^2), \qquad \check{\nu}(\zeta) = -\frac{1}{2\pi}\log(1-|\check{r}(-k_0)|^2), \qquad \epsilon = \frac{1-k_0}{2},\\
&\phi(\zeta,z) = \Phi\left(\zeta, k_0 - \frac{\epsilon z}{\rho}\right) = \overline{\Phi\left(\zeta, \overline{-k_0 + \frac{\epsilon z}{\rho}}\right)}\\
&\qquad = -\frac{48i\tilde{k}_0^3}{(1+4\tilde{k}_0^2)^2} + \frac{i}{2}z^2 + O(z^3), \qquad z \to 0.
\end{aligned} \tag{5.12}$$

Indeed, by adding a number of arcs on which $v = I$, we can ensure that Γ is a Carleson jump contour which satisfies (Γ1)-(Γ4). Furthermore, the decay properties of $e^{\pm t\Phi}$ imply that $w = v - I$ satisfies (4.5) and (4.6). The definition (5.9) of v implies that (4.7) and (4.8) are satisfied with

$$\begin{cases}
R_1(\zeta,t,z) = \frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}r(k)z^{2i\nu(\zeta)},\\
R_2(\zeta,t,z) = \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2}\frac{\overline{r(\bar{k})}}{1-r(k)\overline{r(\bar{k})}}z^{-2i\nu(\zeta)},\\
R_3(\zeta,t,z) = \frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}\frac{r(k)}{1-r(k)\overline{r(\bar{k})}}z^{2i\nu(\zeta)},\\
R_4(\zeta,t,z) = \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2}\overline{r(\bar{k})}z^{-2i\nu(\zeta)},
\end{cases}$$

where k and z are related by $k = k_0 - \frac{\epsilon z}{\rho}$, and

$$\begin{cases} \check{R}_1(\zeta,t,z) = \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2}\overline{\check{r}(\bar{k})}z^{2i\check{\nu}(\zeta)}, \\ \check{R}_2(\zeta,t,z) = \frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}\frac{\check{r}(k)}{1-\check{r}(k)\overline{\check{r}(\bar{k})}}z^{-2i\check{\nu}(\zeta)}, \\ \check{R}_3(\zeta,t,z) = \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{\delta(\zeta,k)^2}\frac{\overline{\check{r}(\bar{k})}}{1-\check{r}(k)\overline{\check{r}(\bar{k})}}z^{2i\check{\nu}(\zeta)}, \\ \check{R}_4(\zeta,t,z) = \frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}\check{r}(k)z^{-2i\check{\nu}(\zeta)}, \end{cases}$$

where k and z are related by $k = -k_0 + \frac{\epsilon z}{\rho}$. The definition (5.12) of $\phi(\zeta,z)$ shows that (4.9) and (4.10) hold. The symmetry $\delta(\zeta,k) = 1/\overline{\delta(\zeta,\bar{k})}$ implies that $|\delta(\zeta,\omega k_0)\delta(\zeta,\omega^2 k_0)| = 1$. Hence $|q(\zeta)| = |r(k_0)|$ and $|\check{q}(\zeta)| = |\check{r}(-k_0)|$; this yields (4.12). To establish (4.11), we note that if $k = k_0 - \frac{\epsilon z}{\rho}$, then

$$\begin{aligned} R_1(\zeta,t,z) = {} & \big((k_0^{-1}-k)(k^{-1}-k_0)kk_0\big)^{i\nu}\bigg(\frac{(k_0+k)(k_0^{-1}+k^{-1})}{(k_0^{-1}+k)(k_0+k^{-1})}\bigg)^{i\check{\nu}} \\ & \times \frac{e^{2\chi(\zeta,k)}}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}r(k)\bigg(\frac{\rho}{\epsilon}\bigg)^{2i\nu(\zeta)}, \end{aligned}$$

so that $q(\zeta) = R_1(\zeta,t,0)$. Similarly, if $k = -k_0 + \frac{\epsilon z}{\rho}$, then

$$\begin{aligned} \check{R}_1(\zeta,t,z) = {} & \bigg(\frac{(k_0^{-1}-k)(k_0-k^{-1})}{(k_0-k)(k_0^{-1}-k^{-1})}\bigg)^{-i\nu}\bigg(\frac{1}{(k_0^{-1}+k)(k_0+k^{-1})kk_0}\bigg)^{-i\check{\nu}} \\ & \times \frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2 k)}{e^{2\chi(\zeta,k)}}\overline{\check{r}(\bar{k})}\bigg(\frac{\rho}{\epsilon}\bigg)^{2i\check{\nu}(\zeta)}, \end{aligned}$$

so that $\check{q}(\zeta) = \check{R}_1(\zeta,t,0)$. The inequalities in (4.11) for R_1 and $\check{R}_1$ are now a consequence of standard estimates cf. [12]. The inequalities in (4.11) for $\{R_j,\check{R}_j\}_2^4$ are proved in a similar way. This shows that the conditions of Theorem 4.1 are satisfied.

The conclusion (4.13) of Theorem 4.1 implies that the solution $n(y,t,k)$ of the L^3-RH problem (5.11) satisfies

$$\begin{aligned} n(y,t,K_1) = {} & (1,1,1) + \frac{2\epsilon}{k_0\sqrt{\tau}}\mathrm{Re}\big(\mathcal{F}_1\beta - \bar{\mathcal{F}}_2\check{\beta}, \mathcal{F}_3\beta - \bar{\mathcal{F}}_3\check{\beta}, \mathcal{F}_2\beta - \bar{\mathcal{F}}_1\check{\beta}\big) \\ & + O\big(\epsilon\tau^{-\frac{1+\alpha}{2}}\big), \qquad \tau = t\rho^2 \to \infty, \quad \zeta \in \mathcal{I}, \end{aligned} \tag{5.13}$$

where the error term is uniform with respect to $\zeta \in \mathcal{I}$ and the functions $\{\mathcal{F}_j\}_1^3$, β, $\check{\beta}$ are defined in (4.14) and (4.15).

Remark 5.2. In general, the solution m of the L^3-RH problem (4.4) featured in Theorem 4.1 is different than the function $\mathfrak{m}$ used in this section; the former is regular at the points $\{\varkappa_j\}_1^6$ whereas the latter, in general, is singular at these points. However, by Lemma A.5, this discrepancy disappears when premultiplying by $(1,1,1)$; hence the row vector solution n of the L^3-RH problem (5.11) satisfies (5.13).

Step 5: Find $q(x,t)$ and $u(x,t)$. For $k \in E_{25}$ near K_1 we have

$$\mathfrak{m} = MJ_{19,26}\Delta = P(k)^{-1}D(x,t)^{-1}P(k)\Phi_7 e^{x\mathcal{L}+t\mathcal{Z}}S_1^{-1}S_8 e^{-y\mathcal{L}-t\mathcal{Z}}e^{\nu_0\mathcal{L}}\Delta.$$

Using the identity

$$(1,1,1)P(k)^{-1}D(x,t)^{-1}P(k) = q(x,t)(1,1,1),$$

this gives

$$n(y,t,k) = q(x,t)(1,1,1)\Phi_7(x,t,k)e^{x\mathcal{L}+t\mathcal{Z}}S_1^{-1}(k)S_8(k)e^{-y\mathcal{L}-t\mathcal{Z}}e^{\nu_0\mathcal{L}}\Delta(\zeta,k).$$

Since $\Phi_n(x,t,K_1) = I$, evaluation of this equation at $k = K_1$ yields

$$\begin{aligned} n(y,t,K_1) &= q(x,t)(1,1,1)e^{\mathcal{L}(K_1)(x-y+\nu_0)}\Delta(\zeta,k) \\ &= q(x,t)(e^{y-x-\nu_0}\Delta_{11}(\zeta,K_1), \Delta_{22}(\zeta,K_1), e^{x-y+\nu_0}\Delta_{33}(\zeta,K_1)). \end{aligned}$$

Hence, by (5.13),[2]

$$\frac{\Delta_{11}(\zeta,K_1)\Delta_{33}(\zeta,K_1)}{\Delta_{22}^2(\zeta,K_1)} = \frac{n_1(\zeta,t,K_1)n_3(\zeta,t,K_1)}{n_2^2(\zeta,t,K_1)} = 1 + O(\epsilon\tau^{-1/2}), \qquad \tau\to\infty.$$

Fixing $\zeta \in \mathcal{I}$ on the left-hand side of this equation and letting $t \to \infty$, we deduce that $\Delta_{11}(\zeta,K_1)\Delta_{33}(\zeta,K_1) = \Delta_{22}^2(\zeta,K_1)$ for $\zeta \in \mathcal{I}$. Proceeding as in the proof of Proposition 4.2 of [4], we infer that $\Delta_{22}(\zeta,K_1) = 1$ and $|r(k_0)| = |\check{r}(-k_0)|$ for all $\zeta \in \mathcal{I}$. It follows that $\nu(\zeta) = \check{\nu}(\zeta)$ for $\zeta \in \mathcal{I}$. The equation

$$\delta(\zeta,k) = e^{\frac{1}{4\pi i}\int_{k_0}^{k_0^{-1}} \log(1-|r(s)|^2)\left(\frac{1}{s-k} - \frac{1}{s+k} - \frac{1}{s-k^{-1}} + \frac{1}{s+k^{-1}}\right)ds},$$

now shows that $\delta(\zeta,k) = \delta(\zeta,-k)^{-1}$ and $\chi(\zeta,k) = -\chi(\zeta,-k)$. In particular, $q(\zeta) = \check{q}(\zeta)e^{i(\arg\check{r}(-k_0)+\arg r(k_0))}$.

A computation shows that

$$\begin{aligned} \arg q(\zeta) &= 2\nu\log\left(\frac{2}{k_0+k_0^{-1}}\right) + \chi_0(\zeta) + \arg r(k_0) + 2\nu\log\left(\frac{\rho}{\epsilon}(k_0^{-1}-k_0)k_0\right) \\ &\quad - \frac{\nu}{2}\log\left|\frac{(k_0^{-1}-k)(k_0-k^{-1})}{(k_0-k)(k_0^{-1}-k^{-1})}\frac{(k_0+k)(k_0^{-1}+k^{-1})}{(k_0^{-1}+k)(k_0+k^{-1})}\right|_{k=\omega k_0} \\ &\quad - \frac{\nu}{2}\log\left|\frac{(k_0^{-1}-k)(k_0-k^{-1})}{(k_0-k)(k_0^{-1}-k^{-1})}\frac{(k_0+k)(k_0^{-1}+k^{-1})}{(k_0^{-1}+k)(k_0+k^{-1})}\right|_{k=\omega^2 k_0} \\ &= \chi_0(\zeta) + \arg r(k_0) - \nu\log Y. \end{aligned} \tag{5.14}$$

where

$$\chi_0(\zeta) = \mathrm{Im}\,(2\chi(\zeta,k_0) - \chi(\zeta,\omega k_0) - \chi(\zeta,\omega^2 k_0))$$

and the function $Y = Y(\zeta)$ is defined by

$$Y(\zeta) = \frac{(4\tilde{k}_0^2+1)^2(4\tilde{k}_0^2+3)}{576\tilde{k}_0^3(3-4\tilde{k}_0^2)}.$$

Equations (5.13) and (5.14) yield

$$\begin{aligned} n_1(y,t,K_1) &= 1 + \frac{d_1}{\sqrt{t}}\mathrm{Re}\left(\left(\mathcal{F}_1 e^{-i\arg r(k_0)} - \bar{\mathcal{F}}_2 e^{i\arg\check{r}(-k_0)}\right)e^{i(d_2 t - \nu\log t + d_3)}\right) \\ &\quad + O(\epsilon\tau^{-\frac{1+\alpha}{2}}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} q(x,t) = n_2(y,t,K_1) &= 1 + \frac{d_1}{\sqrt{t}}\mathrm{Re}\left(\left(\mathcal{F}_3 e^{-i\arg r(k_0)} - \bar{\mathcal{F}}_3 e^{i\arg\check{r}(-k_0)}\right)e^{i(d_2 t - \nu\log t + d_3)}\right) \\ &\quad + O(\epsilon\tau^{-\frac{1+\alpha}{2}}), \qquad \tau\to\infty, \quad \zeta\in\mathcal{I}, \end{aligned} \tag{5.16}$$

[2]All error terms of the form $O(\cdot)$ in the remainder of the proof are uniform with respect to $\zeta \in \mathcal{I}$.

where the functions $d_j = d_j(\zeta)$, $j = 1, 2, 3$, are defined by

$$d_1 = \frac{2\epsilon\sqrt{\nu}}{k_0\rho}, \qquad d_2 = \frac{48\tilde{k}_0^3}{(1+4\tilde{k}_0^2)^2}, \qquad d_3 = \frac{\pi}{4} - \chi_0 + \nu\log Y + \arg\Gamma(i\nu).$$

It can be seen from the proof of Theorem 5.1 that thanks to the uniform decay and smooth dependence of the jump matrix v on t, the asymptotic formula (4.13) can be differentiated in time without affecting the error term. Hence equations (5.15) and (5.16) together with the fact that

$$y - x = \log\left(\frac{n_1(y,t,K_1)}{n_2(y,t,K_1)}\right) - \log\Delta_{11}(\zeta, K_1) + \nu_0, \tag{5.17}$$

yield

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\bigg|_{y \text{ fixed}}(x-y) \\ &= \frac{\partial}{\partial t}\bigg|_{y \text{ fixed}}\frac{d_1}{\sqrt{t}}\mathrm{Re}\left(fe^{i(d_2t-\nu\log t+d_3)}\right) + \frac{\partial}{\partial t}\bigg|_{y \text{ fixed}}\log\Delta_{11}(\zeta,K_1) + O(\epsilon\tau^{-\frac{1+\alpha}{2}}), \end{aligned} \tag{5.18}$$

where

$$f(\zeta) = (\mathcal{F}_3 - \mathcal{F}_1)e^{-i\arg r(k_0)} - (\bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_2)e^{i\arg\check{r}(-k_0)}.$$

As $\zeta \to 3^-$, we have the expansions

$$\begin{aligned} &\epsilon = \frac{1}{12}(3-\zeta)^{\frac{1}{2}} + O(3-\zeta), && k_0 = 1 - \frac{(3-\zeta)^{\frac{1}{2}}}{6} + O(3-\zeta), \\ &\tilde{k}_0 = \frac{(3-\zeta)^{\frac{1}{2}}}{6} + O((3-\zeta)^{\frac{3}{2}}), && \rho = \frac{(3-\zeta)^{\frac{3}{4}}}{\sqrt{6}} + O((3-\zeta)^{\frac{5}{4}}), \\ &d_1 = \frac{\sqrt{\nu}(3-\zeta)^{-\frac{1}{4}}}{\sqrt{6}} + O((3-\zeta)^{\frac{3}{4}}), && d_2 = \frac{2(3-\zeta)^{\frac{3}{2}}}{9} + O((3-\zeta)^{\frac{5}{2}}), \\ &\mathcal{F}_1 = \frac{3+\sqrt{3}}{2} + O((3-\zeta)^{\frac{1}{2}}), && \mathcal{F}_2 = \frac{-3+\sqrt{3}}{2} + O((3-\zeta)^{\frac{1}{2}}), \\ &\mathcal{F}_3 = -\sqrt{3} + O((3-\zeta)^{\frac{1}{2}}), && Y = \frac{3(3-\zeta)^{-\frac{3}{2}}}{8} + O((3-\zeta)^{-\frac{1}{2}}). \end{aligned}$$

We deduce that there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} &c_1\epsilon^{3/2} < \rho(\zeta) < c_2\epsilon^{3/2}, \qquad \nu'(\zeta) < c_2\epsilon^{-1}, \qquad f'(\zeta) < c_2\epsilon^{-1}, \\ &d_1'(\zeta) < c_2\epsilon^{-\frac{5}{2}}, \qquad d_3'(\zeta) < c_2\epsilon^{-2}, \end{aligned} \tag{5.19}$$

for all $\zeta \in \mathcal{I}$. Moreover, since

$$\Delta_{11}(\zeta, K_1) = e^{\frac{3}{2\pi}\int_{k_0}^{k_0^{-1}}\log(1-|r(s)|^2)\frac{1+s^4}{1+s^6}ds}, \tag{5.20}$$

we obtain the estimates

$$c_1\epsilon < |1 - \Delta_{11}(\zeta,K_1)| < c_2\epsilon, \quad c_1\epsilon^{-1} < \left|\partial_\zeta\Delta_{11}(\zeta,K_1)\right| < c_2\epsilon^{-1}, \qquad \zeta \in \mathcal{I},$$

which show that

$$\left|\frac{\partial}{\partial t}\bigg|_{y \text{ fixed}}\log\Delta_{11}(\zeta,K_1)\right| = \left|\frac{\zeta\partial_\zeta\Delta_{11}(\zeta,K_1)}{t\Delta_{11}(\zeta,K_1)}\right| < C\epsilon^{-1}t^{-1} = O(\epsilon^2\tau^{-1}) \tag{5.21}$$

as $\tau \to \infty$. Using the estimates (5.19) and (5.21) together with the identities

$$\frac{\partial \zeta}{\partial t}\bigg|_{y \text{ fixed}} = -\frac{\zeta}{t} \quad \text{and} \quad d_2 + t\frac{\partial d_2}{\partial t} = \frac{6\tilde{k}_0}{1+4\tilde{k}_0^2},$$

equation (5.18) yields

$$u(x,t) = \frac{d_1}{\sqrt{t}} \frac{6\tilde{k}_0}{1+4\tilde{k}_0^2} \mathrm{Re}\left(i f e^{i(d_2 t - \nu \log t + d_3)}\right) + O(\epsilon \tau^{-\frac{1+\alpha}{2}}), \qquad \tau \to \infty, \quad \zeta \in \mathcal{I}. \tag{5.22}$$

Since $\frac{d}{dx} = q\frac{d}{dy}$ we find $u_{xx} = u_{yy} + O(\epsilon\tau^{-\frac{1+\alpha}{2}})$. Using that $\frac{1}{1+4\tilde{k}_0^2}(1 + (\frac{dd_2}{d\zeta})^2) = 1$, we conclude that

$$u - u_{xx} + 1 = 1 + \frac{d_1}{\sqrt{t}} 6\tilde{k}_0 \mathrm{Re}\left(i f e^{i(d_2 t - \nu \log t + d_3)}\right) + O(\epsilon \tau^{-\frac{1+\alpha}{2}}). \tag{5.23}$$

Substituting (5.16) and (5.23) into the relation $q^3 = u - u_{xx} + 1$, the terms of $O(t^{-1/2})$ yield

$$\big(6\tilde{k}_0 i(\mathcal{F}_3 - \mathcal{F}_1) - 3\mathcal{F}_3\big) - \big(6\tilde{k}_0 i(\bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_2) - 3\bar{\mathcal{F}}_3\big) e^{i \arg \check{r}(-k_0) + i \arg r(k_0)} = 0,$$

that is,

$$e^{i \arg \check{r}(-k_0) + i \arg r(k_0)} = \frac{6\tilde{k}_0 i(\mathcal{F}_3 - \mathcal{F}_1) - 3\mathcal{F}_3}{6\tilde{k}_0 i(\bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_2) - 3\bar{\mathcal{F}}_3} = \frac{1-\omega k_0^2}{k_0^2 - \omega}. \tag{5.24}$$

Using (5.24) and the identity

$$\mathcal{F}_3 - \bar{\mathcal{F}}_3 \frac{1-\omega k_0^2}{k_0^2-\omega} = \frac{2\sqrt{3}\tilde{k}_0\sqrt{4\tilde{k}_0^2+3}}{4\tilde{k}_0^2+1} e^{-i\arctan\left(\sqrt{3}\frac{1+k_0^2}{1-k_0^2}\right)}$$

in (5.16), we find

$$q(x,t) = 1 + \frac{c_1(\zeta)}{\sqrt{t}} \cos\big(c_2(\zeta) t - \nu(\zeta)\log(t) + c_3(\zeta)\big) + O(\epsilon\tau^{-\frac{1+\alpha}{2}}), \qquad \tau \to \infty, \quad \zeta \in \mathcal{I}, \tag{5.25}$$

where

$$c_1(\zeta) = \frac{1-k_0^2}{1+k_0^2}\sqrt{\frac{(3+4\tilde{k}_0^2)(1+4\tilde{k}_0^2)\nu}{3\tilde{k}_0 - 4\tilde{k}_0^3}}, \qquad c_2(\zeta) = \frac{48\tilde{k}_0^3}{(1+4\tilde{k}_0^2)^2},$$

$$\begin{aligned} c_3(\zeta) &= d_3 - \arg r(k_0) - \arctan\left(\sqrt{3}\frac{1+k_0^2}{1-k_0^2}\right) \\ &= \frac{\pi}{4} - \chi_0(\zeta) + \nu\log Y + \arg\Gamma(i\nu) - \arg r(k_0) - \arctan\left(\sqrt{3}\frac{1+k_0^2}{1-k_0^2}\right). \end{aligned}$$

Similarly, using (5.24) in (5.22), we find

$$\begin{aligned} u(x,t) = {}& \frac{6\tilde{k}_0 d_1}{(1+4\tilde{k}_0^2)\sqrt{t}} \mathrm{Re}\left(i\left(\mathcal{F}_3 - \mathcal{F}_1 - (\bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_2)\frac{1-\omega k_0^2}{k_0^2-\omega}\right) e^{i(d_2 t - \nu\ln t + d_3 - \arg r(k_0))}\right) \\ & + O(\epsilon\tau^{-\frac{1+\alpha}{2}}). \end{aligned}$$

In view of the identity

$$\mathcal{F}_3 - \mathcal{F}_1 - (\bar{\mathcal{F}}_3 - \bar{\mathcal{F}}_2)\frac{1-\omega k_0^2}{k_0^2-\omega} = -\frac{i\sqrt{3}\sqrt{4\tilde{k}_0^2+3}}{4\tilde{k}_0^2+1} e^{-i\arctan\left(\sqrt{3}\frac{1+k_0^2}{1-k_0^2}\right)},$$

this yields

$$u(x,t) = \frac{3c_1(\zeta)}{(1+4\tilde{k}_0^2(\zeta))\sqrt{t}} \cos\big(c_2(\zeta)t - \nu(\zeta)\log t + c_3(\zeta)\big) + O(\epsilon\tau^{-\frac{1+\alpha}{2}}),$$
$$\tau \to \infty, \quad \zeta \in \mathcal{I}, \tag{5.26}$$

uniformly with respect to $\zeta \in \mathcal{I}$.

Step 6: Replace ζ with ξ. In the last step of the proof, we show that, up to a phase shift, $\zeta = y/t$ can be replaced with $\xi = x/t$ in the asymptotic formulas (5.25) and (5.26) without affecting the error term. For clarity, we reinsert the dependence on ζ of the functions $k_0(\zeta)$, $\tilde{k}_0(\zeta)$, and $\epsilon(\zeta)$.

By (5.15), (5.16), and (5.17), we have

$$y - x = -\log \Delta_{11}(\zeta, K_1) + \nu_0 + O(\epsilon(\zeta)\tau^{-1/2}).$$

Hence

$$\zeta - \xi = \frac{c(\zeta)}{t} + O(\epsilon(\zeta)t^{-1}\tau^{-\frac{1}{2}}) = O((3-\zeta)^{\frac{1}{2}}t^{-1}) = O((3-\xi)^{\frac{1}{2}}t^{-1}), \qquad \tau \to \infty.$$

The asymptotic sector $\{\zeta \in \mathcal{I}, \tau \to \infty\}$ is equivalent to $\{\zeta \in \mathcal{I}, t(3-\zeta)^{\frac{3}{2}} \to \infty\}$ and hence also to $\{\xi \in \mathcal{I}, t(3-\xi)^{\frac{3}{2}} \to \infty\}$. If $\alpha \in \mathbb{R}$ and $g(\zeta)$ is a smooth function such that $|g'(\zeta)| < C(3-\zeta)^{\alpha}$ for all $\zeta \in \mathcal{I}$, then

$$\begin{aligned}
|g(\xi) - g(\zeta)| &= \left|\int_{\zeta}^{\xi} g'(\eta)d\eta\right| < C|(3-\xi)^{\alpha+1} - (3-\zeta)^{\alpha+1}| \\
&< C|\xi - \zeta| \max\big\{(3-\zeta)^{\alpha}, (3-\xi)^{\alpha}\big\} < C|\xi-\zeta|(3-\xi)^{\alpha} \\
&= O\big((3-\xi)^{\alpha+\frac{1}{2}}t^{-1}\big), \qquad \tau \to \infty, \quad \zeta \in \mathcal{I}.
\end{aligned}$$

The estimates

$$\begin{aligned}
&|c_1'(\zeta)| < C(3-\zeta)^{-\frac{3}{4}}, \qquad |c_2'(\zeta)| < C(3-\zeta)^{\frac{1}{2}}, \qquad |c_3'(\zeta)| < C(3-\zeta)^{-1}, \\
&|\nu'(\zeta)| < C(3-\zeta)^{-\frac{1}{2}}, \qquad |\epsilon'(\zeta)| < C(3-\zeta)^{-\frac{1}{2}}, \qquad |\tilde{k}_0'(\zeta)| < C(3-\zeta)^{-\frac{1}{2}},
\end{aligned}$$

therefore imply

$$\begin{aligned}
&|c_1(\xi) - c_1(\zeta)| = O\big((3-\xi)^{-\frac{1}{4}}t^{-1}\big), \qquad |c_2(\xi) - c_2(\zeta)| = O\big((3-\xi)t^{-1}\big), \\
&|c_3(\xi) - c_3(\zeta)| = O\big((3-\xi)^{-\frac{1}{2}}t^{-1}\big), \qquad |\nu(\xi) - \nu(\zeta)| = O\big(t^{-1}\big), \\
&|\epsilon(\xi) - \epsilon(\zeta)| = O\big(t^{-1}\big), \qquad |\tilde{k}_0(\xi) - \tilde{k}_0(\zeta)| = O\big(t^{-1}\big), \qquad \tau \to \infty, \quad \zeta \in \mathcal{I}.
\end{aligned} \tag{5.27}$$

On the other hand, the identity

$$c_2'(\zeta) = \frac{dc_2}{d\tilde{k}_0}\bigg/\frac{d\zeta}{d\tilde{k}_0} = -2\tilde{k}_0(\zeta), \qquad \zeta \in \mathcal{I},$$

and the estimate

$$|c_2''(\zeta)| < C(3-\zeta)^{-\frac{1}{2}}, \qquad \zeta \in \mathcal{I},$$

imply

$$\begin{aligned}
c_2(\zeta) - c_2(\xi) &= -2\tilde{k}_0(\xi)(\zeta - \xi) + O((3-\xi)^{-\frac{1}{2}}(\zeta-\xi)^2) \\
&= 2\tilde{k}_0(\xi)t^{-1}\log \Delta_{11}(\zeta, K_1) + O((3-\xi)^{\frac{1}{4}}t^{-\frac{3}{2}}).
\end{aligned} \tag{5.28}$$

Since

$$\chi(\zeta, k) = \frac{1}{4\pi i}\int_{k_0}^{\frac{1}{k_0}} \log\left(\frac{1-|r(s)|^2}{1-|r(k_0)|^2}\right)\left(\frac{1}{s-k} - \frac{1}{s+k} - \frac{1}{s-k^{-1}} + \frac{1}{s+k^{-1}}\right)ds,$$

we see that $\chi_0(\xi)$ can be expressed as in (5.3). Employing equations (5.20), (5.27), and (5.28), the asymptotic formulas (5.1) and (5.2) follow from (5.25) and (5.26), respectively.

Remark 5.3. Substituting the asymptotic formula (5.2) for $u(x,t)$ into (1.1), we can verify explicitly that the DP equation is satisfied to leading order in the similarity region. Indeed, by (5.2), the nonlinear terms in (1.1) are easily seen to be of order $O(\epsilon\tau^{-\frac{1+\alpha}{2}})$ as $\tau\to\infty$, whereas the linear terms satisfy

$$(u-u_{xx})_t+3u_x=O(\epsilon\tau^{-\frac{1+\alpha}{2}})$$

as a consequence of the identity

$$\left(b_2-\xi\frac{\partial b_2}{\partial\xi}\right)\left(1+\left(\frac{\partial b_2}{\partial\xi}\right)^2\right)+3\frac{db_2}{d\xi}=0.$$

Remark 5.4. The main contributions to the asymptotic formula (5.2) come from the critical points $\omega^j k_0^{\pm1}$ located on the lines $\omega^j\mathbb{R}$, $j=0,1,2$, only. On the other hand, the same is true for the asymptotics of the solution of the whole line problem [4]. Therefore, the structure of the asymptotics for the whole line and half-line problems is the same, the only difference being in the determination of $r(k)$ which, in turn, determines ν (h_0 in the notation of [4]).

Following [4], these contributions can be determined by parametrizing the neighborhood of $k=k_0$ using the rescaled spectral parameter $z=\frac{\rho_{\text{new}}}{\epsilon}(\tilde{k}-\tilde{k}_0)$, where (recall that $\Phi(\zeta,k)=F(\tilde{k}(k))$)

$$\phi_{\text{new}}(\zeta,z)=F(\tilde{k}_0+\frac{\epsilon}{\rho_{\text{new}}}z)$$

$$\rho_{\text{new}}=\sqrt{-i\epsilon^2F''(\tilde{k}_0)}=\epsilon\sqrt{\frac{48\tilde{k}_0(3-4\tilde{k}_0^2)}{(1+4\tilde{k}_0^2)^3}},$$

and we denote quantities defined using this rescaled spectral parameter by the subscript *new*. It follows that (4.7) and (4.8) are satisfied with

$$\begin{cases} R_1^{\text{new}}(\zeta,t,z)=\frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2k)}r(k)z^{2i\nu(\zeta)},\\ R_2^{\text{new}}(\zeta,t,z)=\frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2k)}{\delta(\zeta,k)^2}\frac{\overline{r(\bar{k})}}{1-r(k)\overline{r(\bar{k})}}z^{-2i\nu(\zeta)},\\ R_3^{\text{new}}(\zeta,t,z)=\frac{\delta(\zeta,k)^2}{\delta(\zeta,\omega k)\delta(\zeta,\omega^2k)}\frac{r(k)}{1-r(k)\overline{r(\bar{k})}}z^{2i\nu(\zeta)},\\ R_4^{\text{new}}(\zeta,t,z)=\frac{\delta(\zeta,\omega k)\delta(\zeta,\omega^2k)}{\delta(\zeta,k)^2}\overline{r(\bar{k})}z^{-2i\nu(\zeta)}, \end{cases}$$

where k and z are related by $\tilde{k}=\tilde{k}_0+\frac{\epsilon z}{\rho_{\text{new}}}$. Hence

$$R_1^{\text{new}}=R_1\left(\frac{k_0+k^{-1}}{k_0+k_0^{-1}}\right)^{2i\nu},$$

and so $q^{\text{new}}=q$. The proof of Theorem 4.1 proceeds in the same way as before except that equation (4.37) is replaced with

$$\begin{aligned} m_0(\zeta,t,k)^{-1}&=\mathcal{C}D(\zeta,t)m^X\left(q(\zeta),\frac{\sqrt{\tau_{\text{new}}}}{\epsilon}(\tilde{k}-\tilde{k}_0)\right)^{-1}D(\zeta,t)^{-1}\mathcal{C}\\ &=I+\frac{B(\zeta,t)}{\sqrt{\tau_{\text{new}}}(\tilde{k}-\tilde{k}_0)}+O(\tau^{-1}),\qquad \tau_{\text{new}}\to\infty,\quad \zeta\in\mathcal{I},\quad |k-k_0|=\epsilon. \end{aligned} \tag{5.29}$$

Hence, the contribution from the critical point at k_0 to $m(\zeta,t,K_1)$ is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|k-k_0|=\epsilon}\frac{\hat{\mu}(\zeta,t,k)(m_0(\zeta,t,k)^{-1}-I)dk}{k-K_1}\\
&=\frac{1}{2\pi i}\int_{|k-k_0|=\epsilon}\frac{(m_0(\zeta,t,k)^{-1}-I)dk}{k-K_1}+\frac{1}{2\pi i}\int_{|k-k_0|=\epsilon}\frac{(\hat{\mu}(\zeta,t,k)-I)(m_0(\zeta,t,k)^{-1}-I)dk}{k-K_1}\\
&=\frac{1}{2\pi i}\int_{|k-k_0|=\epsilon}\frac{B(\zeta,t)}{\sqrt{\tau_{\text{new}}}(\tilde{k}-\tilde{k}_0)}\frac{dk}{k-K_1}\\
&\quad+O(\epsilon\tau_{\text{new}}^{-1})+O\big(\|\hat{\mu}-I\|_{\dot{L}^3(\hat{\Gamma})}\|m_0^{-1}-I\|_{L^{3/2}(|k-k_0|=\epsilon)}\big)\\
&=\frac{1}{2\pi i}\frac{B(\zeta,t)}{\sqrt{\tau_{\text{new}}}}\int_{|k-k_0|=\epsilon}\frac{2dk}{(\frac{1}{k}-\frac{1}{k_0}-(k-k_0))(k-K_1)}+O(\epsilon\tau_{\text{new}}^{-\frac{1+\alpha}{2}})\\
&=-\frac{2}{\sqrt{\tau_{\text{new}}}(\frac{1}{k_0^2}+1)}\frac{B(\zeta,t)}{k_0-K_1}+O(\epsilon\tau_{\text{new}}^{-\frac{1+\alpha}{2}}).
\end{aligned}\tag{5.30}$$

This leads to the same formula for the asymptotics of $u(x,t)$ as above because $\frac{1}{2}(1+k_0^{-2})\rho_{\text{new}}=\rho$.

Taking into account the correspondence of notations (ν, k_0, and $\tilde{k}_0$ in the current paper correspond to h_0, κ_0, and p_0 in [4], respectively), formulas (5.2) and (5.3) actually correct the coefficients c_1 and c_4 in the corresponding asymptotic formula (4.1) in [4], where the contribution of the critical points to $m(K_1)$ was treated incorrectly.

Appendix A. L^p-Riemann-Hilbert problems

Since the jump contour for the RH problem associated with equation (1.1) on the half-line has nontransversal intersection points (see Figure 1), special care has to be taken when defining the notion of an L^p-RH problem. We will follow [16] where a theory of L^p-RH problems with jumps across Carleson contours is developed using generalized Smirnoff classes.

Let $\mathcal{J}$ denote the collection of all subsets Γ of the Riemann sphere $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ such that Γ is homeomorphic to the unit circle and

$$\sup_{z\in\Gamma\cap\mathbb{C}}\sup_{r>0}\frac{|\Gamma\cap D(z,r)|}{r}<\infty,\tag{A.1}$$

where $D(z,r)$ denotes the disk of radius r centered at z. Curves satisfying (A.1) are called *Carleson curves*; all contours considered in this paper are Carleson. Let $p\in[1,\infty)$. If D is the bounded component of $\hat{\mathbb{C}}\setminus\Gamma$ where $\Gamma\in\mathcal{J}$ and $\infty\notin\Gamma$, then a function f analytic in D belongs to the *Smirnoff class* $E^p(D)$ if there exists a sequence of rectifiable Jordan curves $\{C_n\}_1^\infty$ in D, tending to the boundary in the sense that C_n eventually surrounds each compact subdomain of D, such that

$$\sup_{n\geq 1}\int_{C_n}|f(z)|^p|dz|<\infty.\tag{A.2}$$

If D is a subset of $\hat{\mathbb{C}}$ bounded by an arbitrary curve in $\mathcal{J}$, $E^p(D)$ is defined as the set of functions f analytic in D for which $f\circ\varphi^{-1}\in E^p(\varphi(D))$, where $\varphi(z)=\frac{1}{z-z_0}$ and z_0 is any point in $\mathbb{C}\setminus\bar{D}$. The subspace of $E^p(D)$ consisting of all functions $f\in E^p(D)$ such that $zf(z)\in E^p(D)$ is denoted by $\dot{E}^p(D)$. If $D=D_1\cup\cdots\cup D_n$ is the union of a finite number of disjoint subsets of $\hat{\mathbb{C}}$ each of which is bounded by a curve in $\mathcal{J}$, then $E^p(D)$ and $\dot{E}^p(D)$ denote the set of functions f analytic in D such that $f|_{D_j}\in E^p(D_j)$

and $f|_{D_j} \in \dot{E}^p(D_j)$ for each j, respectively. We define $E^\infty(D)$ as the space of bounded analytic functions on D.

A *Carleson jump contour* is a connected subset Γ of $\hat{\mathbb{C}}$ such that:

(a) $\Gamma \cap \mathbb{C}$ is the union of finitely many oriented arcs[3] each pair of which have at most endpoints in common.
(b) $\hat{\mathbb{C}} \setminus \Gamma$ is the union of two disjoint open sets D_+ and D_- each of which has a finite number of simply connected components in $\hat{\mathbb{C}}$.
(c) Γ is the positively oriented boundary of D_+ and the negatively oriented boundary of D_-, i.e. $\Gamma = \partial D_+ = -\partial D_-$.
(d) If $\{D_j^+\}_1^n$ and $\{D_j^-\}_1^m$ are the components of D_+ and D_-, then $\partial D_j^+ \in \mathcal{J}$ for $j = 1, \dots, n$, and $\partial D_j^- \in \mathcal{J}$ for $j = 1, \dots, m$.

We henceforth make the following assumptions: (a) $p \in (1, \infty)$ and $n \geq 1$ is an integer, (b) $\Gamma = \partial D_+ = -\partial D_-$ is a Carleson jump contour, and (c) $v : \Gamma \to GL(n, \mathbb{C})$ is an $n \times n$-matrix valued function. We define $\dot{L}^p(\Gamma)$ as the set of all measurable functions on Γ such that $|z - z_0|^{1-\frac{2}{p}} h(z) \in L^p(\Gamma)$ for some (and hence all) $z_0 \in \mathbb{C} \setminus \Gamma$. If $f \in \dot{E}^p(D_+)$ or $f \in \dot{E}^p(D_-)$, the nontangential limits of $f(z)$ as z approaches the boundary exist a.e. on Γ and the boundary function belongs to $\dot{L}^p(\Gamma)$. Let $D = D_+ \cup D_-$. A *solution of the L^p-RH problem determined by* (Γ, v) is an $n \times n$-matrix valued function $m \in I + \dot{E}^p(D)$ such that the nontangential boundary values $m_\pm$ satisfy $m_+ = m_- v$ a.e. on Γ.

Lemma A.1 (Uniqueness)**.** *Suppose $1 \leq n \leq p$ and $\det v = 1$ a.e. on Γ. If the solution of the L^p-RH problem determined by (Γ, v) exists, then it is unique and has unit determinant.*

If $h \in \dot{L}^p(\Gamma)$, then the Cauchy transform $\mathcal{C}h$ defined by

$$(\mathcal{C}h)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{h(s)}{s - z} ds, \qquad z \in \mathbb{C} \setminus \Gamma, \tag{A.3}$$

satisfies $\mathcal{C}h \in \dot{E}^p(D)$. We denote the nontangential boundary values of $\mathcal{C}h$ from the left and right sides of Γ by $\mathcal{C}_+ h$ and $\mathcal{C}_- h$ respectively. We henceforth fix a point $z_0 \in \mathbb{C} \setminus \Gamma$ and turn $\dot{L}^p(\Gamma)$ into a Banach space with the norm

$$\|h\|_{\dot{L}^p(\Gamma)} := \| |\cdot - z_0|^{1-\frac{2}{p}} h \|_{L^p(\Gamma)}. \tag{A.4}$$

Then $\mathcal{C}_+$ and $\mathcal{C}_-$ are bounded operators on $\dot{L}^p(\Gamma)$ and $\mathcal{C}_+ - \mathcal{C}_- = I$. Given a function $w \in \dot{L}^p(\Gamma) \cap L^\infty(\Gamma)$, we define $\mathcal{C}_w : \dot{L}^p(\Gamma) + L^\infty(\Gamma) \to \dot{L}^p(\Gamma)$ by $\mathcal{C}_w(h) = \mathcal{C}_-(hw)$. Then

$$\|\mathcal{C}_w\|_{\mathcal{B}(\dot{L}^p(\Gamma))} \leq C \|w\|_{L^\infty(\Gamma)}. \tag{A.5}$$

where $C := \|\mathcal{C}_-\|_{\mathcal{B}(\dot{L}^p(\Gamma))} < \infty$ and $\mathcal{B}(\dot{L}^p(\Gamma))$ denotes the space of bounded linear operators on $\dot{L}^p(\Gamma)$.

Lemma A.2. *Suppose $w := v - I \in \dot{L}^p(\Gamma) \cap L^\infty(\Gamma)$. If $m \in I + \dot{E}^p(D)$ satisfies the L^p-RH problem determined by (Γ, v), then $\mu = m_- \in I + \dot{L}^p(\Gamma)$ satisfies*

$$\mu - I = \mathcal{C}_w(\mu) \quad in \quad \dot{L}^p(\Gamma). \tag{A.6}$$

Conversely, if $\mu \in I + \dot{L}^p(\Gamma)$ satisfies (A.6), then $m = I + \mathcal{C}(\mu w) \in I + \dot{E}^p(D)$ satisfies the L^p-RH problem determined by (Γ, v).

[3]A subset $\Gamma \subset \mathbb{C}$ is an *arc* if it is homeomorphic to a connected subset of the real line which contains at least two distinct points.

Lemma A.3. *Let D be a subset of $\hat{\mathbb{C}}$ bounded by a curve $\Gamma \in \mathcal{J}$ and let $f : D \to \mathbb{C}$ be an analytic function. If there exist curves $\{C_n\}_1^\infty \subset \mathcal{J}$ in D, tending to Γ in the sense that C_n eventually surrounds each compact subset of $D \subset \hat{\mathbb{C}}$, such that*

$$\sup_{n\geq 1} \int_{C_n} |z - z_0|^{p-2} |f(z)|^p |dz| < \infty, \tag{A.7}$$

then $f \in \dot{E}^p(D)$.

Lemma A.4 (Contour deformation)**.** *Let $\gamma \in \mathcal{J}$. Suppose that, reversing the orientation on a subcontour if necessary, $\hat{\Gamma} = \Gamma \cup \gamma$ is a Carleson jump contour. Let B_+ and B_- be the two components of $\hat{\mathbb{C}} \setminus \gamma$. Let $\hat{D}_\pm$ be the open sets such that $\hat{\mathbb{C}} \setminus \hat{\Gamma} = \hat{D}_+ \cup \hat{D}_-$ and $\partial \hat{D}_+ = -\partial \hat{D}_- = \hat{\Gamma}$. Let $\hat{D} = \hat{D}_+ \cup \hat{D}_-$. Let γ_+ and γ_- be the parts of γ that belong to the boundary of $\hat{D}_+ \cap B_+$ and $\hat{D}_- \cap B_+$, respectively. Suppose $v : \Gamma \to GL(n, \mathbb{C})$. Suppose $m_0 : \hat{D} \cap B_+ \to GL(n, \mathbb{C})$ satisfies*

$$m_0, m_0^{-1} \in I + \dot{E}^p(\hat{D} \cap B_+) \cap E^\infty(\hat{D} \cap B_+). \tag{A.8}$$

Define $\hat{v} : \hat{\Gamma} \to GL(n, \mathbb{C})$ by

$$\hat{v} = \begin{cases} m_{0-} v m_{0+}^{-1} & \text{on} \quad \Gamma \cap B_+, \\ m_{0+}^{-1} & \text{on} \quad \gamma_+, \\ m_{0-} & \text{on} \quad \gamma_-, \\ v & \text{on} \quad \Gamma \cap B_-. \end{cases}$$

Let m and $\hat{m}$ be related by

$$\hat{m} = \begin{cases} m m_0^{-1} & \text{on} \quad \hat{D} \cap B_+, \\ m & \text{on} \quad \hat{D} \cap B_-. \end{cases} \tag{A.9}$$

Then $m(z)$ satisfies the $\dot{L}^p$-RH problem determined by (Γ, v) if and only if $\hat{m}(z)$ satisfies the $\dot{L}^p$-RH problem determined by $(\hat{\Gamma}, \hat{v})$.

Proofs of the above statements can be found in [16]. We will also need the following uniqueness result for row vector solutions.

Lemma A.5. *Suppose $1 \leq n \leq p$ and $\det v = 1$ a.e. on Γ. Suppose the L^p-RH problem determined by (Γ, v) has a unique solution m. If n is a row vector solution of the L^p-RH problem determined by (Γ, v) in the sense that $n \in (1, 1, \dots, 1) + \dot{E}^p(D)$ and $n_+ = n_- v$ a.e. on Γ, then $n = (1, 1, \dots, 1)m$.*

Proof. Let $\hat{m}$ denote the $n \times n$-matrix valued function obtained from m by replacing the first row with the row vector n. Then the $n \times n$-matrix valued function $\tilde{m} \in I + \dot{E}^p(D)$ defined by

$$\tilde{m} = \begin{pmatrix} 1 & -1 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \hat{m}$$

satisfies the L^p-RH problem determined by (Γ, v). Hence $\tilde{m} = m$ by Lemma A.1. Consequently, $n = (1, 1, \dots, 1)\tilde{m} = (1, 1, \dots, 1)m$. ∎

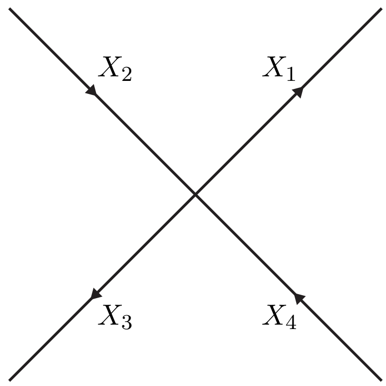

Figure 8. *The contour* $X = X_1 \cup \cdots \cup X_4$.

Appendix B. The solution on a cross

Consider the cross $X = X_1 \cup \cdots \cup X_4 \subset \hat{\mathbb{C}}$ where

$$\begin{aligned} X_1 &= \big\{ue^{\frac{i\pi}{4}} \,\big|\, 0 \le u \le \infty\big\}, & X_2 &= \big\{ue^{\frac{3i\pi}{4}} \,\big|\, 0 \le u \le \infty\big\}, \\ X_3 &= \big\{ue^{-\frac{3i\pi}{4}} \,\big|\, 0 \le u \le \infty\big\}, & X_4 &= \big\{ue^{-\frac{i\pi}{4}} \,\big|\, 0 \le u \le \infty\big\}, \end{aligned} \tag{B.1}$$

and X is oriented as in Figure 8. Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and define the function $\nu : \mathbb{D} \to (0, \infty)$ by $\nu(q) = -\frac{1}{2\pi} \log(1 - |q|^2)$. We consider the following family of L^3-RH problems parametrized by $q \in \mathbb{D}$:

$$\begin{cases} m^X(q, \cdot) \in I + \dot{E}^3(\hat{\mathbb{C}} \setminus X), \\ m_+^X(q, z) = m_-^X(q, z) v^X(q, z) \quad \text{for a.e. } z \in X, \end{cases} \tag{B.2}$$

where the jump matrix $v^X(q, z)$ is defined by[4]

$$v^X(q,z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ qz^{-2i\nu(q)} e^{\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_1, \\ \begin{pmatrix} 1 & -\frac{\bar{q}}{1-|q|^2} z^{2i\nu(q)} e^{-\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ -\frac{q}{1-|q|^2} z^{-2i\nu(q)} e^{\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_3, \\ \begin{pmatrix} 1 & \bar{q} z^{2i\nu(q)} e^{-\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_4. \end{cases}$$

[4]Throughout the paper, complex powers and logarithms are defined using the principal branch: If $z, a \in \mathbb{C}$ and $z \neq 0$, then $\log z := \log|z| + i\mathrm{Arg}\, z$ and $z^a := e^{a \log z}$, where $\mathrm{Arg}\, z \in (-\pi, \pi]$ denotes the principal value of $\arg z$.

Theorem B.1. *The L^3-RH problem (B.2) has a unique solution $m^X(q,z)$ for each $q \in \mathbb{D}$. This solution satisfies*

$$m^X(q,z) = I + \frac{i}{z}\begin{pmatrix} 0 & -\beta^X(q) & 0 \\ \overline{\beta^X(q)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O\Big(\frac{1}{z^2}\Big), \qquad z \to \infty, \quad q \in \mathbb{D}, \tag{B.3}$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ *and q in compact subsets of $\mathbb{D}$, and the function $\beta^X(q)$ is defined by*

$$\beta^X(q) = \sqrt{\nu(q)}e^{i\left(\frac{\pi}{4} - \arg q + \arg\Gamma(i\nu(q))\right)}, \qquad q \in \mathbb{D}. \tag{B.4}$$

Moreover, $m^X(q,\cdot) \in E^\infty(\hat{\mathbb{C}} \setminus X)$.

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